Transformations

Volume 6
Issue 1 *Winter 2020*

Article 1

11-30-2020

Winter 2020

TRANSFORMATIONS

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VOLUME 6 ISSUE 1 WINTER 2020

Transformations:
A Journal of the Florida Association of Mathematics Teacher Educators

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From the President's Desk…

Dear Mathematics Educators:

I am excited that the Winter issue of the Transformation Journal is ready for your use. This Journal is made available online through NSUWorks. I encourage you to submit your research articles so that we can share with the mathematics educators around the country. I also invite you to nominate a colleague or self-nominate to serve on our Board so that we can help make a difference in the K-22 mathematics education community in the State of Florida and throughout the country.

As an affiliate of the Florida Council of Teachers of Mathematics (FCTM), I am looking forward to achieving the following goals over the next two years:

1. Annual FAMTE Conference to promote the improvement of Florida's mathematics instructional programs and to promote cooperation and communication among the teachers of mathematics and mathematics teacher educators in Florida.
2. FAMTE Board represented by at least one K-12 Mathematics Teacher educators.
3. Promote scholarly publications.

With Warm Regards,

Hui Fang Huang "Angie" Su,
FAMTE President and Editor of
Transformation

Transformations

Manuscript 1029

Transformative Learning: An Approach to Understand Participatory Action Research

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Bal Luitel

Abstract

Transformative learning is to observe one's experience that makes one conscious of one's knowledge and one's changing view in the learning community. This paper presents reflection on transformation in learning both the theory and practice of participatory action research as experienced through field visits and interacting with the participants. Critical reflective journal entries and discussions on workshops provided source of data. The purpose of this paper is to provide understanding of participatory action research from traditional conventional research and how reflection helped unfold the ideologies of participatory action research. Following the cycle of participatory action research, authors explored the need of the participation of all the members of this study. Through collaborative learning, social constructive learning, experiential learning as well as transformative learning, this paper explains ideologies of participatory action research, which are co-construction of knowledge, change in attitude to bring transformation in practice, and empowerment of participants.

Keywords: Participatory Action Research, transformative learning, co-construction of knowledge

Transformative learning: An approach to understand participatory action research

Introduction

Participatory action research (PAR) is an ecosystem of its own kind because it follows an iterative cycles of planning, action, observation and reflection (Walker, 1993) that brings change in both the researcher and the researched. For this reason, participatory action research is open inquiry that involves not only the researcher in the process of inquiry but also the participants to find problems and to take action to solve the problem. With an inquiry of understanding PAR, authors dwelled into doing PAR trying to learn and improve one's practice of doing participatory research which brought transformation within.

To bring change in the quality of education in the public schools of Dapcha, Kavre under *RAAA* (which means transformation) project and to accomplish the fourth goal of *The 2030 Agenda for Sustainable Development* which is to "ensure inclusive and equitable quality education and promote lifelong learning opportunity" (United Nations, 2015, p. 21), this project looks after five schools of Namobuddha Municipality, Nepal. Moreover, the vision of *RAAA* is to bring change in education, health and livelihood of the school community. Based on Taylor and Cranton (2012), the research questions that guided this paper are "how am I building and refuting the transformative theory?" and "what transformation occurred in the researcher during the exploration of the participatory action research?"

Thus, this paper presents reflection on transformative learning, transformation in doing research from traditional academic research to participatory action research; transformation in understanding theory of PAR and how to practice PAR in the field; and learning through experience gained from field visits, interacting with participants, colleagues and workshops. The purpose of this paper is to criticize positivistic view of academic research and to explore changing view from doing traditional research to doing participatory action research. This paper contributes to the literature by providing evidences of transformation of researchers and co-researchers by practicing participatory action research that revealed ideologies of *participatory action research*. This paper presents three aspects: transformative learning process; exploration of academic research versus participatory action research; and ideologies of PAR that evolved.

Transformative learning

Learning depends on one's existence as well as experience perceived from the interaction of individual with the environment and interaction between individuals. Mezirow (2009) explains transformative learning as "a metacognitive process of *reassessing reasons* supporting our problematic meaning perspectives" (p. 96). The purpose of transformative learning is to bring change in the perspective to allow individual to participate in critical reflective discourses to acquire reflective judgment. For instance, during one of the initial workshops that *RAAA* conducted, the teachers' participation was superficial and seemed forceful on the first few days. Getting into the issues of school practice and challenging the teachers in school's practical issues was one way to get teachers involved in reflective discourse. One of issues that teachers faced was not conducting extra-curricular activities in the school. Based on evidences from the previous year, the teachers discussed the reasons for not having extra-curricular activities in the school. When the teachers were involved in the practical issues, they started looking for resources that can solve those issues. Identifying enough resources, finally the teachers decided to do extra-curricular activities and prepared a plan of conducting such extra-curricular activities every Friday, which is evident on their calendar. Even though the workshop was initiated by the researchers, the decision of doing extra-curricular activities was purely voluntary through reflective judgment based on the context, subjectivity, variety of sources, as well as reasonable inquiry (King & Kitchener, 2004), which is still in effect as the teachers felt accountable for conducting such activities.

One's prior knowledge plays an important role in transformation. By becoming conscious of one's own understanding and evaluating one's own prior knowledge, one changes views. The steps that Mezirow (1997) explained for transformative learning are elaboration of existing point of view, establishment of new point of view, transform the point of view, and becoming aware and critically reflecting biases that one has. Even though learning is not linear and does not follow this linear path, Mezirow's principles of transformative learning helped unfold transformation. For instance, in the previous example, the workshop helped participants and researchers elaborate existing point of view by getting involved in discussion that established new point of view. When participants and researchers added new schemas through discourse, becoming aware of one's bias and critically reflecting on one's bias, transformation of the existing point of view occurred. From Mezirow's point of view, for the transformation to happen, there is a need to shift in the frame of reference, which does not happen until and unless, one is comfortable in one's comfort zone. This workshop provided evidence that participatory action research is a kind of research that brings change when co-researchers (the participants) are involved. This event brought transformation not only in the participants but also on researchers. The researchers had the understanding that participation of the teachers in action makes participants accountable of their *action*, which makes *the action* a success.

Even though transformative learning is individualistic in nature that interprets one's own transformation through self-realization, supporting and refuting one's own frame of reference and habit of mind, transformation also happens through discourses, outsider provoking one's understanding, and challenging one's existing point of view. By becoming conscious of one's own knowledge and of one's changing view, one can experience transformation. Transformative learning differs from individual to individual as experience that one has, critical reflection that one does, capacity to engage in dialogue, and context are different for every person (Taylor, 2012). This experience that transforms a life is even different when the researcher is a woman (English & Irving, 2012) as the circumstances are different for different gender. Mezirow (1997) further explained "self-reflection can lead to significant personal transformations" (p.7). Transformation in frames of reference takes place through critical reflection and transformation of habit of mind, or they may result from an accretion of transformations in points of view. In short, transformation needs to come from within.

Change in Attitude: Academic Research versus Participatory Action Research

With the term 'research', one has an understanding that it is done in the universities where researchers are experts in their field. A positivist researcher conducts research with some hypothesis, testify the hypothesis with numbers and come to a conclusion based on the numbers that signifies on knowledge driven by facts. Whereas non-positivist studies people, acknowledging their experiences, culture, norms and values, and thus gathers information from the people to generate theory. For such, researchers go to the field, gather information by observing the participants, asking questions, taking pictures and conducting experiments. Heron and Reason (2001) refer such kind of research as *research 'on' people rather than research 'with' people* where the participants are viewed as information providers, which create hierarchy of power between participants and the researchers.

Academicians usually have such views on conducting research and transformation of such views is one of the focuses of this paper. Researchers' previously held beliefs about research are usually rooted through academia, where researchers decides the problem as well as on methods based on literature, which is purely “theoretical rather than practical” (Heron & Reason, 2001, p. 179). With a thought *participatory action research is just another kind of research*, I (first author) started doing research through *RAAA* project. With first few visits to the field, authors planned out not only the problems of the research but also interventions to make the public school a better one by introducing ICT training, integrating technology in the classroom, planting trees around the school and so on. At the end, the vision of academic research is to publish articles in profitable journals that focus on production of subject-oriented result that never reach the participants.

As university academician, the glass that one wears controls one’s thinking which creates a gap between researcher and the participants creating hierarchy. However, after being involved in the field and practicing participatory approach to research, it became clear that the conventional research does not help local people understand the problem and bring change in their context. Participatory action research is not doing research on the people but with the people involving them on research, deciding about the content, problems, methods as well as interventions. Exploring participatory action research, authors became aware of the misconceptions and beliefs that I (first author) have on *research.* Becoming aware of the misconceptions is the first step of transformation; reflecting critically on the misconceptions is the second step; based on this reflection, stepping back is another step; and through another step that is changing one’s action as well as praxis brought transformation, which unfolded the ideologies of *participatory action research* PAR, which are stated below.

Ideologies of Participatory Action Research

Exploring through collaboration with the participants, gaining personal experience, changing one's view by interacting with the participants and the environment through social constructive learning, and critically reflecting on one's own action, experience and learning, some of the ideologies of participatory action research that stood out are: i) As much as interaction is important, rapport building is equally important to get into the community of practice; ii) indigenous knowledge that participants bring make them experts; iii) researchers and participants are co-researchers without anyone of which, the bicycle of PAR does not go long distance; iv) knowledge is co-constructed by interacting with the co-researchers; v) it empowers participants by breaking the power dynamics; vi) and participatory action is dialectical in nature.

The first thing that was evident was the importance of rapport building with the participants. In due course, researchers mimicked the life-style of the school community. For instance, we stayed in a health center; we walked to the school. On the way to the school, we walked with the local people, talked with them; stopped by local tea-shops and drank tea with them. This helped understand the local environment as well as it helped build rapport with the community. Even before the start of the research, we went to the school and stayed in the school premises to make ourselves familiar with the teachers and the school environment and to make them familiar with us. This is unlike academic research, where one does not need to understand the needs of the participants whereas in participatory action research, one cannot know the need of the participants without knowing the participants. Without building trust with the participants, without knowing the environment and without being in the shoes of the participants, intervention was out of question.

Being naïve in the field of *participatory action research*, we were looking for problems that could be corrected through intervention. In other words, we were looking for weakness of the school community. Participants have been living in the community and utilizing local resources and knew the resources of the community. They have been benefitting from their local knowledge from a long time. Realization that participants are the experts of the community came to us in a long run. Thus, being involved in participatory action research, we learned to respect indigenous people and their wisdom (Rahnema, 1990). It also became evident that doing participatory action research means doing research and intervention being culturally responsive and acknowledging local wisdom.

In due course, during one of the workshops, researchers and teachers came together to discuss strengths, weaknesses, needs and issues. Talking about weaknesses brought discomfort among the participants as no one wants to disclose weaknesses. We came to know that the best way to involve people into action is to engage them in their own inquiry. The beauty of PAR revealed was that it challenges the participants to think beyond the boundary to find not only the problems or weaknesses, but also the solutions to the problem through mutual collaboration. PAR helped share power with participants as the participants got involved in finding problems, identify resources in their surrounding, produce knowledge, empower themselves and involve in finding solutions to their problems (Baum, MacDougall, & Smith, 2006), which also increased the researcher's knowledge directly (Chesler, 1991). Thus, learning from each other, PAR encouraged the researched (participants) to become the researcher or co-researcher.

Furthermore, PAR is dialectical in nature which means reality is understood in contradiction of two opposite views. Being dialectic with the participants provided space for both the researcher and the participants (researched) to view the problem not in isolation but as a part of the whole (Roberts, 2003). Through collaborative discussion, the participants expressed themselves and got involved in the research, thus became co-researchers. By acknowledging the views of the co-researchers, they felt their voices being heard, which also empowered the co-researchers. By getting involved in the PAR, the participants became not only valuable audiences but eventually experts. For example, many teachers did not know how to use the computer. However, after getting involved in PAR, teachers felt the need of using the resources they have; they learned to use the computer from few teachers who knew how to use it. Now, the teachers use computer lab and posted pictures in the Facebook.

During these periods of field visit and being engaged with the co-researcher, I (the first author) learned to back up and not impose my understanding to the participants. I challenged their thinking, triggered their thoughts, guided them but I did not provide solutions to the problem. During one of the workshops, teachers formed two groups: one group involved in preparing activity for the teachers and one group for the students. They prepared activity for students and teachers. Being involved in PAR and having an understanding that the activity was not applicable to their context, I could not directly tell them nor could I tell them what to do. My other option was to inquire to make the teachers think realistically where I had to challenge them, not provide solution but guide. My role changed from an academic solution provider to a facilitator and I became a part of field (Chesler, 1991). Thus, teachers got focused on practical aspect and prepared activity that was relevant to them and one of the activities was learning to use computer, which eventually happened in the initiation of the teachers. Being a teacher, an educator, we tend to correct the mistakes of the students, provide solutions to the students to develop skills and insight. By not imposing my thoughts or solutions to the problems and being a part of the field, my point of view changed that brought transformation on my research practice. PAR brought change in my role from an academic researcher to a facilitator.

In the process of finding problems and solutions to the problems, the participants were involved in dialogue. Coming into consensus, co-researchers agree on solutions and thus co-construct knowledge. Through participation of both the researcher and the researched and both becoming equally engaged in the research, production of knowledge is possible through continuous dialogue between the researcher and the researched (Kong, 2017).Thus the participants and the researchers became the coauthors of knowledge produced. PAR disrupts the professional monopoly in knowledge creation process and helps in knowledge creation through co-construction of knowledge. Through constant interaction with the co-researchers and reflecting on the action, the strength of PAR is to co-construct knowledge and empowerment of the co-researchers.

Participation is the key to participatory action research. By participating the object/researched in the process of action research in the meaning making process, PAR blurs the line of researcher and researched, empowering the researched. This further disrupts the conventional power hierarchy between the researcher and the researched. The participants when involved PAR, transformation of co-researchers is the ultimatum. By following bottom-up approach, PAR empowers the participants.

Cutting through the power dynamics in this turmoil of researcher versus researched was the hardest. A university degree differentiates researcher and researched providing power to the researcher as knowledge is power, which creates hierarchy between the researcher and the researched. When university graduates identify problems of the community, the participants believe it to be their problem (Rahnema, 1990). Because of the cultural lens that one wears, sometimes it is hard to notice this hegemony. When the indigenous people's knowledge is acknowledged and accepted, it provides power to the participants to co-construct knowledge, which ultimately breaks the power dynamics between the researcher and the researched.

Conclusions

The PAR is a complex system as a human body with different organs working simultaneously for the proper functioning of the body. Looking at each fragment separately does not give full picture of the whole system; however, different components provide strength to the PAR. The holistic approach is what makes PAR a complete and complex system.

Through constant interaction with the participants with whom collaborative study is done, through self-reflection as well as critiques from my colleagues and exploring through dialogues there was a shift in the habit of mind or frame of reference. Furthermore, through collaboration with community of practice (in the field), sharing knowledge with colleagues, transformation happened.

In conclusion, this transformative learning helped me unfold my prior beliefs about research and some ideologies of PAR. The participation of both the researcher and the researched, getting involved in the dialogue, critiquing oneself brought transformation (Kong, 2017).PAR disrupts conventional power hierarchy, empowers co-researchers which happened as a result of collaborative learning. PAR allowed me and my participants to socially construct knowledge, which I learned through experience. However, there are more ideologies to be discovered and more transformation to happen as the petals of PAR unravel to bloom into a beautiful flower.

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Using GeoGebra, Photography, and Vocabulary to Teach Mathematics while Aiding our ESOL Populations

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Using GeoGebra, Photography, and Vocabulary to Teach Mathematics while Aiding our ESOL Populations

Abstract

This paper shares many important examples of using real world photography and GeoGebra software in teaching math to all students including the ESOL population. ESOL learners can benefit immensely from the use of real-world photography imported into GeoGebra software because new vocabulary and math concepts are concretized. Students are often excited and motivated by photography and technology. Thus, this paper will demonstrate how mathematics teachers can insert photos into GeoGebra Software and then explore math concepts relevant in the photographs, i.e. shapes, symmetry, measurement, fractions, parabolas, etc. By using software and a visual approach to teaching math with photographs, many of the Common Core and state math standards are met while also helping all students including the ESOL learners. GeoGebra software is available as a free download and many school districts are adopting this software in their classrooms as it is a powerful tool for teaching and connecting geometry, measurement, and algebra. The authors advocate teaching mathematics using authentic and relevant materials such as photography and technology in meeting the needs of all students, particularly the ESOL learners who have limited English language skills. Undoubtedly, using photographs is one way to motivate ESOL learners to learn mathematics and English simultaneously.

Keywords: Teaching Mathematics, GeoGebra, Photography, ESOL, Visuals, Vocabulary

Introduction

"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater." And *"...that is why I think a photograph can be kind."*

-Albert Einstein

"Every child is an artist. The problem is how to remain an artist once he grows up."

-Pablo Picasso

In this high-tech and globally competitive STEM driven world it is becoming more and more important that all citizens be confident in their ability to do mathematics. Knowledge of mathematics is an important skill necessary to succeed in today's world. All students deserve equal access to learning math and teachers must make the effort to ensure this. The National Council of Teachers of Mathematics (NCTM, 2000) in their revised and updated standards document specifically identified "Equity" as their first principle for school mathematics. "Excellence in mathematics education requires-high expectations and strong support for all students" (p. 11). NCTM believes that "Equity requires accommodating differences to help everyone learn mathematics" (p. 13). The NCTM has taken a prominent stand that as educators we must take an "equity for all students" approach to teaching mathematics. All students have the right to learn math and feel confident in their ability to do math. Teachers shoulder the responsibility of ensuring that "mathematics can and will be learned by all students" (NCTM, 2000, p. 13).

This paper includes strategies for teaching mathematics to reach all students using photography in GeoGebra along with some English as a Second Language (ESOL) best practices. These teaching strategies may be effective with Exceptional Education (ESE), English Language Learners (ELL), and/or mainstream students. With ESE learners, the use of the multi-modal approach that incorporates the multiple intelligence caters to students' short attention span, as they are not expected to only sit still to learn the materials. English Language Learners (ELL) often need specially designed instruction in English such as the Sheltered Instruction Observation Protocol (SIOP). The strategies and approaches in this paper could help mainstream teachers support ELLs' language acquisition while teaching them the content areas such as Mathematics by emphasizing vocabulary through the use of visuals (Furner, Yahya, & Duffy, 2005). According to Heera Kang (2016), providing ELLs access to the same level of math rigor as other students is an essential aspect of supporting their overall academic success. When considering instructional solutions inclusive of ELLs–particularly technology-based solutions–here are some features to look for:

- Presents math problems and concepts visually
- Allows interactive, self-directed exploration
- Delivers scaffolded, mastery-based learning
- Introduces mathematical language in strategic intervals
- Provides data and real-time, informative feedback so players can monitor progress and adjust their solutions.

In reality, these strategies really are just best practice for the teaching of mathematics in general.

Using Photographs in GeoGebra Math Tools to Teach Vocabulary

English language learners, especially the beginners, rely on materials with less print and more visuals such as pictures to facilitate their comprehension of the concepts they are learning. Mathematical difficulties of ELLs appear to be related to the language demands of mathematics tasks because of linguistic as well as nonlinguistic processing constraints. (Alt, Arizmendi & Beal, 2014) Most times, ELLs already have understood the math concepts in their L1; all they need to do is to re-label these concepts in English. Another approach in teaching vocabulary to ELLs is through the use of Semiotics. Research has shown that semiotics plays a very vital role in the field of language learning and teaching. According to Senel (2007), semiotics provides a practical teaching/learning process by using body language, pictures, visuals, film-strips, video, photography, etc. With the help of semiotics, language learning becomes more productive and exciting. Qadha and Mahdi (2019) investigated the effect of semiotics on learning abstract words. Fifty-five Arab learners of English as a foreign language (EFL)participated in their study and participants were assigned into three groups. The first group was taught abstract words using semiotics. The second group was taught concrete words using semiotics. The third group was taught the same words using a traditional way, i.e., without semiotics. Results of the post-test indicated that participants in semiotics groups (either concrete or abstract) outscored the participants who did not use semiotics to learn new words. The study concluded that semiotics is a useful tool to enhance learning new words. Also, semiotics can be more helpful in learning concrete words than abstract words. Other semiotics studies have also examined digital computer games for second language acquisition especially vocabulary. Aghlara and Tamjid (2011) examined the effect of using a digital computer game on preschoolers' vocabulary gain; they noticed that children in the experimental group did better than in the control group. Their study discussed the positive effects of using digital computer games on vocabulary gain at the preschool level. Likewise, Segers and Forhoeven (2003) conducted a study in which they studied the impact of using a computer on preschoolers' vocabulary learning. The study concluded that vocabulary training through computer had positive effects on preschool children's vocabulary learning. In addition, Silverman and Hines (2009) conducted a study in which they tried to make a comparison between the classical and multimedia-enhanced read-aloud vocabulary instructions concerning their effects on vocabulary gain of English language learners (ELLs) and non-English language learners (non-ELLs). The study came up with results showing that there was a positive effect for ELLs on a researcher- designed measure and a measure of general vocabulary knowledge. With many studies supporting the notion that teaching vocabulary to ELLs is enhanced with the use of technology, teachers can facilitate ELLs' learning of vocabulary by using technology in teaching math as well. For instance, teachers can point out all the math concepts and vocabulary through the use of GeoGebra math tools. Real objects embedded in the photos can be placed in GeoGebra math tools. This can reinforce the math ideas and concepts visually and enable ELLs to simultaneously learn general English and mathematics vocabulary. It is important to utilize concrete objects that students can use in hands-on activities; this not only makes the comprehension of abstract math concepts more comprehensible, but also fun while connections to the representational models are made in the photos that students can import into the GeoGebra software [See the Figures 1-11 below].

Using Photographs in GeoGebra to Teach Mathematics to the Culturally Diverse Students

Especially for the teaching of students from diverse background, teachers can research the math strategies used by their students from different cultures. For instance, Chinese students may be familiar with the use of abacus to do their calculations. Teachers can perhaps ask these students to show the others in class how abacus is used. Honoring and recognizing students' individual ways of solving math problems will boost their self-esteem because students would feel that they too have something to contribute to the learning process despite their English language constraints. Even students with learning disabilities in the area of math can be powerful teachers to other students when they explain how they learned to accomplish specific skills or concepts (Mastropieri, et al, 2001, p.23). In addition, teachers can prompt students to talk about their experience in learning some of the math concepts in their country. By importing photographs into the GeoGebra, ELL can share photos of their cultural background to explore the math concepts and in doing so learn the English vocabulary that is depicted within these photos.

By capitalizing on students' prior knowledge and using photos to teach math, teachers who are empathetic to their students' needs and backgrounds, bridge the new knowledge to the old, making learning new math concepts more manageable for these students. In some cases, by understanding how students solve problems, teachers can troubleshoot or fine tune the individual student's process and make them more efficient learners, and by connecting the math concepts they learn to the world around them through photos makes the process easier for the ELLs.

Using Photography in Mathematics to Apply Problems to Daily Life Situations

Creative teachers can use a variety of ways to coach students when applying problems to daily life situations especially when they make it meaningful by incorporating photos within the math lesson. For example, teachers can use realia such as restaurant take-out menus to teach students multiplication and division. Not only do such activities involve students with real life situations they also create a fun learning environment for students. Such environment will also promote English language acquisition for ELLs. Krashen's (1985) metaphoric use of affective filter in his affective filter hypothesis reinforces the idea that teachers can lower students' "affective filter" by fostering a spirit of mutual respect, high expectations, and cooperative learning. A group project that students can do now in this pandemic is to go on virtual tours of Museums or Monuments. Students can be given some mathematical problems to solve based on what they have seen on this virtual tour. By using real world practice activities, the goal of generalizing skills learned in class to their lives becomes more attainable. When children see and relate things to real life it makes the learning of math much more meaningful [Refer to the model figures of photography in GeoGebra with vocabulary, below].

Using Drawings and Visualizations to teach Mathematical Concepts

The Natural Approach (Krashen, 1985; Terrell, 1981) is used extensively when teaching ELLs. One of the four principles of this approach is that the teacher understands that the new ELLs go through a silent period before they begin to speak English. One of the subsequent strategies of the Natural Approach is to allow students who are at the early developmental stage of their language acquisition process to use drawings and symbols in solving some of the math problems. The same techniques are employed with students with learning disabilities to allow them to process auditory information before making a verbal response. In fact, as a comprehension strategy, teachers can use students' drawings and verbal rehearsals as testimony of their understanding of math concepts. This approach can alleviate frustrations for both teachers and students. Students can use the GeoGebra software and by inserting photos into the GeoGebra they can then draw on the photo and employ several math functions such as measuring, plotting points, identifying vertices and at the same time learning general English vocabulary and technical Math terminologies. Additionally, students can be encouraged to explore and create drawings and designs using the geometry tools within GeoGebra to discover their artistic talents as they learn mathematics.

Using Technology and Mathematics Software such as GeoGebra to make Math Representational

Today the Internet and computer software such as GeoGebra are now being used as an instructional tool to explore, investigate, problem-solve, interact, reflect, reason, communicate, and learn many concepts that are in U.S. School curricula. Students are able to take virtual tours of places like the Bronx Zoo, the White House, the Louvre Museum, as well as access information from NASA, the United Nations, etc. The number of websites and educational software designed for teachers and students with the intent to teach concepts is becoming infinite. GeoGebra can be used right on the computer or downloaded free and used even off line. There are many websites today like funschool.com or www.funbrain.com which are ideal for both teachers and students to teach/learn a multitude of math and reading/language concepts of K-12 and software like *Mathblaster* or *Tesselmania* can really make the learning of mathematics dynamic. GeoGebra software is one of the most diverse math tools out there now and it is free where students can explore and create tessellating pieces of art and so much more.

Ameis and Ebenezer (2000) wrote a book called *Mathematics on the Internet* where they provide resources and suggestions for teaching mathematics via the Internet. The book connects many math K-12 concepts to many websites that can be used to help teach these concepts. Parents of home-schoolers too can greatly benefit from the use of the Internet as a means to learn via Internet Field Trips. The Internet has a definite role to play in reforming traditional teaching. By using educational software like GeoGebra and the Internet (technology) and photography, all students, especially ELLs, can now learn in ways that are challenging yet more exciting. The Internet too provides students with the tools to use the computer to access information and become independent life-long learners in an age that is increasingly dependent on technology to survive a complex multicultural world.

Educators today should advocate the use of emerging technologies such as the GeoGebra which demonstrates how photos can be imported and used to teach the math concepts[See Figure 1 below on how to insert photos in GeoGebra]. It is critical in the 21st Century that students have an appreciation for math around them and in their everyday life (Gorriz & Vilches 2019). When using visuals, students are often highly motivated to use photos that appeal to their senses while exploring the mathematics within them. Antje, Hannula, & Toivanen (2018) found that using outdoor photography when teaching math has a positive impression on students and their learning of mathematics.

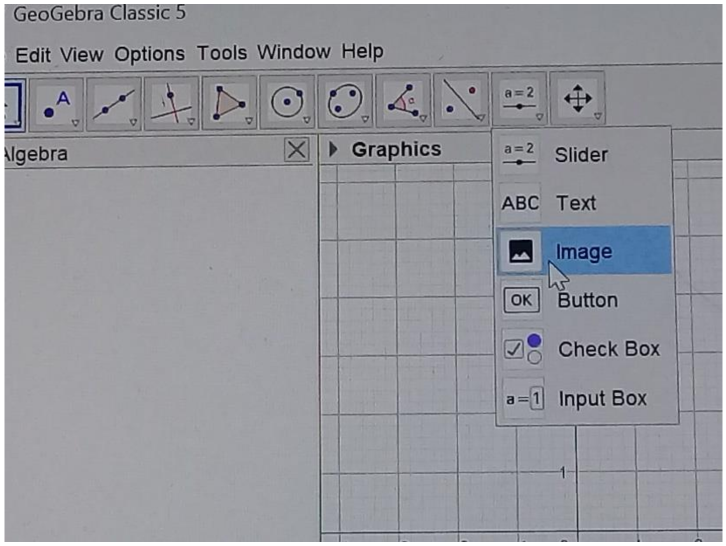

Figure 1. Inserting Photos into the GeoGebra Software

GeoGebra is a multi-platform dynamic mathematics software for all levels of education from elementary through university that joins dynamically geometry, algebra, tables, graphing, spreadsheets, statistics and calculus in one easy-to-use package (Hewson, 2009; Hohenwarter, Hohenwarter, & Lavicza, 2009). This open-source dynamic mathematics software can be downloaded for free and accessed at: http://www.geogebra.org/cms/en/info. Since there are no licensing issues, students and teachers have the freedom to use it both within the classroom and at home. GeoGebra has a large international user and developer community with users from 190 countries is currently translated into 55 languages. This paper advocates the use of GeoGebra, and also how the insertion of photographs and the use GeoGebra tools help teachers to teach important math concepts in grades K-12.

Connections need to be made when we teach mathematics (Zengin, 2019). Munakata and Vaidya (2012) based on their research found that students do not consider mathematics and science to be creative endeavors, although the traditional artistic disciplines rank high in this regard. To address this problem in perception, the authors used photography as a means to encourage students to find the deep-rooted connections between science and mathematics and the arts. The photography project was used in a formal classroom setting as well as an outside activity, i.e. in a more informal setting. Students' interest and motivation were peaked when photography was part of the instructional strategies to teach new material while making meaningful connections to math through photography. Jones (2012) also in her book, *Visualizing Mathematics*, discusses how teachers need to help students visualize and create representations of their math understanding in order to make them excited about the subject.

By using technology and photographs, our young learners are excited to construct and investigate math ideas and learn some history as well with GeoGebra. Students like and enjoy math more by way of photography and the software GeoGebra. Students will see and appreciate the world around them better and this may get them excited about the STEM fields that are so critical today while also instilling a passion for photography. Abraham (2019) contends that when someone loves what they do or have a passion for something such as photography, they can influence others to develop that passion as well.

GeoGebra

While the study of geometry has existed for many millennia, it is within the past twenty years that a shift has occurred in how geometry may be learned through computer-based interactive geometric software. Software programs like GeoGebra allow users to construct interactive representations of points, lines, shapes, and circles. These geometric objects are interactive in that they may be resized and shifted around onscreen through clicking and dragging actions. Furthermore, interactive geometric software like GeoGebra in the K-12 mathematics curriculum has been used at the elementary level, middle school level, and the high school level (Yu & Tawfeeq, 2011) and has proven to be a very effective means for teaching and learning mathematics.

Fahlberg-Stojanovska, & Stojanovski (2009) discovered that using GeoGebra software motivates and helps young people learn at a higher level while exploring and conjecturing as they draw and measure onscreen. Rosen & Hoffman (2009) found that it is very important to integrate both concrete and virtual manipulatives into the primary-age math classroom. Furner & Marinas (2015) found that children easily transition to the abstract when using geometry sketching software by first using geoboards and the software like *Paint* before going directly to the sketching software GeoGebra.

GeoGebra is a great resource and technological tool that when used in the math classroom provides a focus to:

- promote technology as an essential tool for learning mathematics in the 21st century
- integrate the principles and process standards with teaching the content standards
- provide access to all five mathematics content standards for all students
- support learner-centered strategies that address the diverse needs of all learners of mathematics

GeoGebra allows students to actively construct their own understanding of geometry, measurement, and algebra using this technology. Using GeoGebra educators can meet the Common Core Math Standards, students can master many math concepts such as:

- use the polygon and circle tools to draw shapes
- measure angles and distance
- use GeoGebra sliders to adjust values of different problems
- insert images into a file to demonstrate mathematical problem solving
- recognize perimeter as an attribute of plane figures and distinguish between linear and area measures
- reason with shapes and their attributes

Effective mathematics teachers can maximize the potential of GeoGebra to:

- develop students' understanding
- stimulate their interest
- increase their proficiency in mathematics

GeoGebra is a free software, a multi-platform dynamic mathematics software for all levels of education that joins geometry, algebra, tables, graphing, statistics and calculus in one easy-to-use package (Hohenwarter, Hohenwarter, & Lavicza, 2009). GeoGebra has a large international user and developer community with users from 190 countries. The software is currently translated into 55 languages and attracts close to 300,000 downloads per month. It can be downloaded for free and accessed at: http://www.geogebra.org/cms/en/info.

Making Connections when Teaching Mathematics is Important for all Students

Today it is important that educators make important connections between the school curriculum and the real-world to better reach learners. Connections need to be made when we teach mathematics according to Zengin (2019). Connections between the Common Core State Standards and the National Council of Teachers of Mathematics (NCTM) Standards need to meet the challenges of differentiating mathematics instruction in the K-12 classroom. Small (2012) explains two powerful and universal strategies that teachers can use across all math content: Open Questions and Parallel Tasks. Showing teachers how to get started and become experts with these connection strategies. Small also demonstrates more inclusive learning conversations that promote broader student participation and mathematical thinking. Guidance for creating a more inclusive classroom learning community with mathematical talk that engages participants from all levels is important (Small, 2012). While using the technology and covering all the required math standards by making all necessary connections for sound learning, students can work together and use the software like GeoGebra to engage in higher level mathematical understanding.

Andresen & Misfeldt (2010) found that with the Common Core Standards (National Governors Association Center for Best Practices - NGA Center, 2010) in teaching mathematics, teachers need to be trained and learn new mathematics content and technology like GeoGebra in order to be effective in teaching and reaching their students. Knowledge of technology cannot be isolated from the content and good mathematics teaching requires an understanding on how technology is related to the pedagogy and mathematics (Hughes, 2005).

Today, most schools and states are adhering to the Common Core Math Standards (National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO), 2010) found at: http://www.corestandards.org/. A sampling of objectives from the Common Core Standards that can be addressed using the GeoGebra software are presented in many of the photos and examples below [See Figures 2-11]:

Teaching the Common Core Math Standards: Using Photography and GeoGebra

CCSS.MATH.CONTENT.K.G.A.2

Students can correctly name shapes regardless of their orientations or overall size.

CCSS.MATH.CONTENT.2.G.A.1

Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.1 Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

CCSS.MATH.CONTENT.4.G.A.1

Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

CCSS.MATH.CONTENT.6.G.A.3

Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

CCSS.MATH.CONTENT.8.G.B.7

Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

CCSS.MATH.CONTENT.HSG.CO.A.1

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Using Photography while Emphasizing Vocabulary and Math Ideas

Math teachers may want to do a two-minute math starter problems like the following using photos to start and motivate a class lesson (Furner & Marinas, 2020) by using the photos and questions below:

Can you identify where each one of these photos was taken [See Figure 2]?

Can you see some mathematical idea or math vocabulary within the photo?

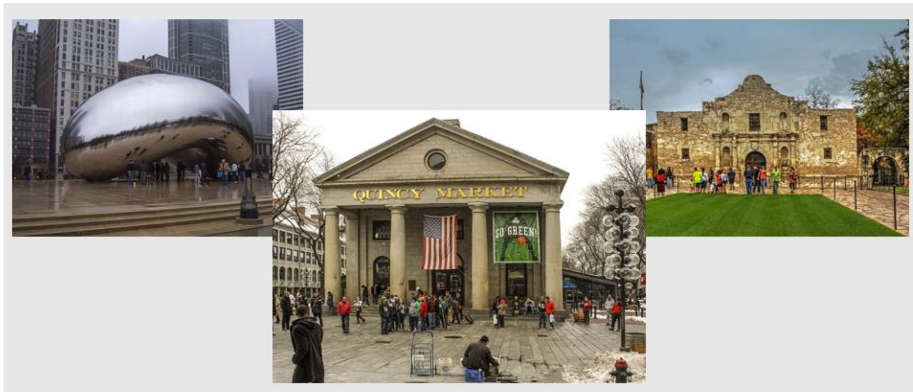

Figure 2. Identify the Photo Locations and Mathematics in them (Photos by Dr. Carol Marinas)

In this example in Figure 2, students may know these famous locations: Chicago, Boston, and San Antonio and they should be able to identify some important vocabulary: **parallel lines, circles, rectangles, symmetry, 3-D solids, angles,** and **repeating patterns**.

Munakata and Vaidya (2012) grounded on their research discovered that students do not consider mathematics and science to be creative endeavors, though the traditional artistic disciplines rank high in this regard. To address this problem in perception, the authors used photography as a means to encourage students to find the deep-rooted connections between science and mathematics and the arts. The photography project was implemented in a formal classroom setting as well as an outside activity, i.e. in a more informal setting. Through the project, it was observed that students' interest and motivation were peaked when photography was part of the instructional approaches to teach new material while making meaningful connections to the math concepts. Bragg & Nicol (2011) contend that students can have success with learning math when they use photos and problem solving together to do the math based on the photos, relating and understanding real-world problems through the use of interactive technology like GeoGebra and connecting them with photography to make important connections in mathematics for students.

Rizzo, del Río, Manceñido, Lavicza, & Houghton (2019) through their research found that by using photograph and GeoGebra and students collaborating to learn math, students were found to be much more motivated to learn the math concepts, and the photography helped students to connect math to the real-world. Through their research and analysis, they found using a combination of three strategies helped to engage learners: relating photography/art, students' surroundings/real-world, and the mathematics content. These three strategies proved to enhance instruction and success in delivering the subject. Portaankorva-Koivisto & Havinga (2019) found that when educators support learning through the use of photography and the visual arts in teaching math, it enhances students' learning and success of understanding mathematics and its values in everyday life. In addition, students can see shared values and ideas in both math and photographs, such as ratio, shapes, and other mathematical and shared ideas.

It is important that we develop a passion for photography in our students for those who are interested. Abraham (2019) contends that when someone loves what they do or have a passion for something like photography, they can influence others to develop that passion as well. Antje, Hannula, & Toivanen (2018) found that when using outdoor photography when teaching math that it had a positive impression on students and their learning of mathematics. It is important we encourage our learners to recognize that geometry and shapes/mathematics that surround us! Students can see this in photos and so much more math can be learned using photographs to teach mathematics. Spring(2020) contends that educators can teach math, English, physics and other subjects using photography. Oswald (2008) reported in her article that teachers can use photography to teach and learn science, math, and writing. Jaqua (2017) contends using photos and even selfies have benefits for students in learning mathematics. Math is already a part of students' lives like selfies are and encouraging students to see the math values when they take photos, valuing the world around them and the math that exists as well, math can then be more real for learners. Encouraging students to take photos may help them develop a passion for photography and help develop a long-term creative outlet and passion for live. Mathematics lends itself well to incorporating photography into its discipline.

Math teachers can ask their students what math concepts can they see in the photo below [See Figure 3]?

Figure 3. Identify Math Concepts in Photo in Orlando taken by Dr. Joseph Furner

Figure 3 offers some vocabulary words in the photo, words like: **points, lines, circles, parallel lines,** to name a few. Oswald (2008) reported on a school-based project where the head teacher, Liz Becks created a project to learn science, math, and writing using photography, Beck is cited in her report saying "There's something in nearly every subject that relates to photography, and it's a topic that appeals to kids" (p. 1). By using photography within instruction educators are appealing to learners and also creating a passion for photography in many of them. Please see Figure 3 for example of parallel lines within a photo of the real-world imported into GeoGebra and the parallel lines drawn on the photo using the GeoGebra tools.

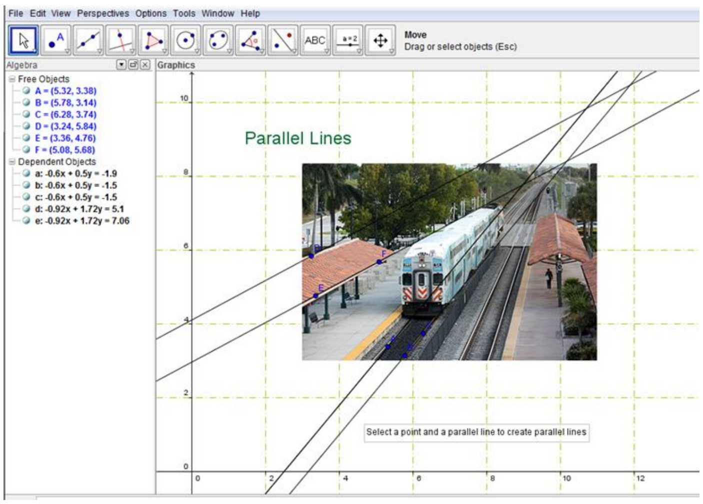

Figure 4. Examples of Parallel Lines in a Photo

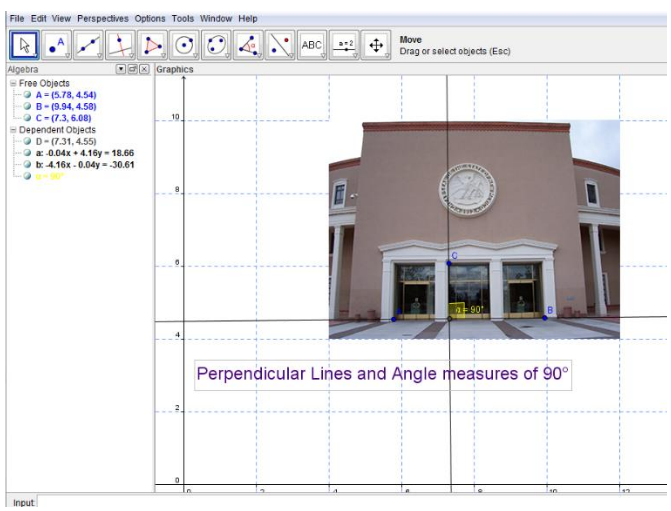

Figure 4. GeoGebra file showing Photo of Perpendicular Lines

Perpendicular lines create **right angles**, **90 degree angles**, and like in the GeoGebra file photo above [See Figure 5] right angles and perpendicular lines are drawn on the photo with the GeoGebra software all allowing students to identify vocabulary and math ideas.

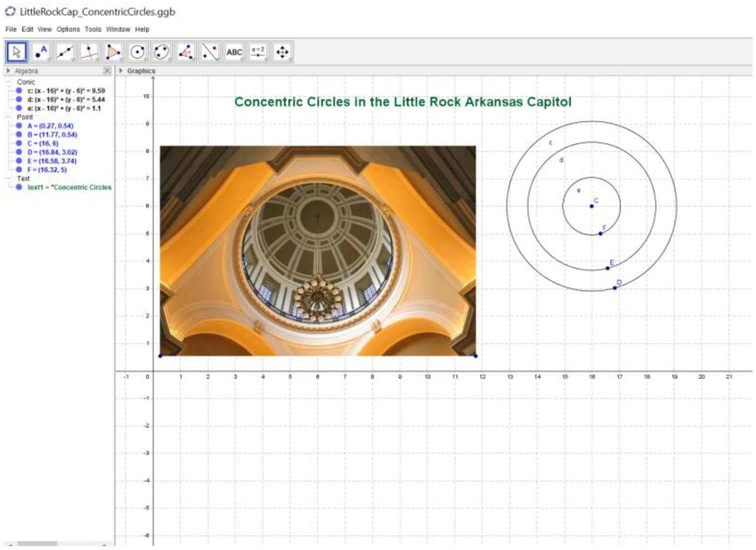

Figure 6. Circles and Concentric Circles in GeoGebra

The photo in Figure 6 was imported into GeoGebra and then students were asked to draw **circles** and **concentric circles** seen in the photo of the Little Rock Capitol building.

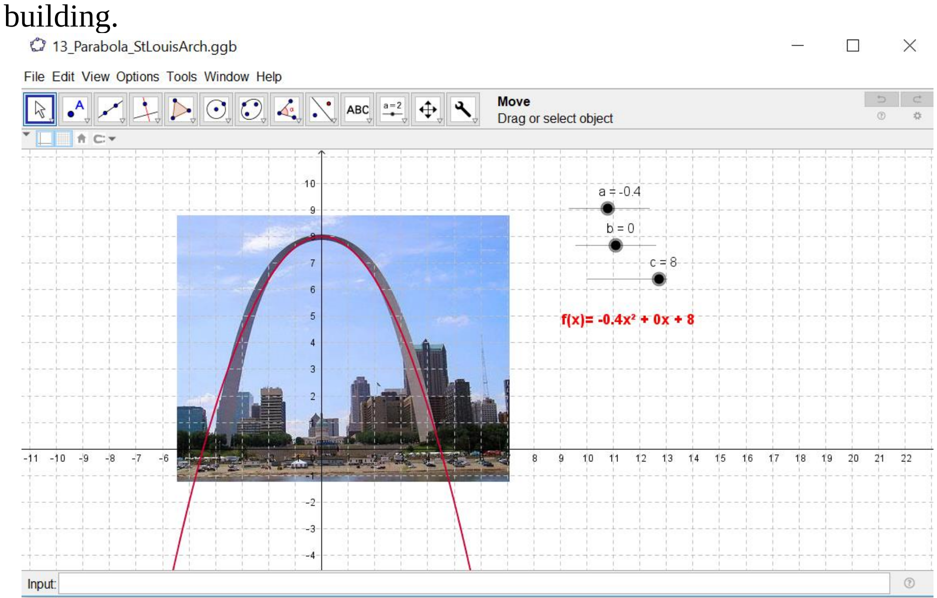

Figure 7. Parabola Examples in the Real World

Parabolas often act as a frame around the central object and can be seen in Figure 7 above with the example of a photo of the St. Louis Arch is like a parabola and students can import the photo into GeoGebra and then try to draw a parabola on it to find its formula.

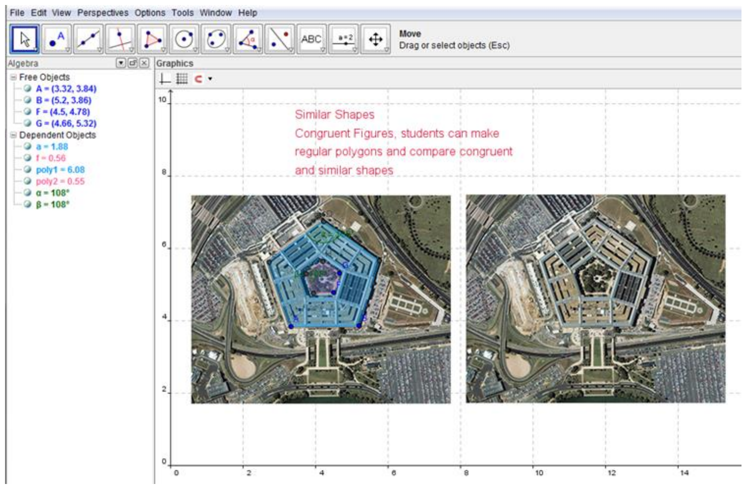

Figure 8. Similar Shapes

Figure 8 depicts an aerial view of the Pentagon photo inserted into GeoGebra and then the students can use the GeoGebra tools to compare **angles** and **pentagon shapes** to see the **similar shapes** of the **interior** and **exterio**r shapes of the Pentagon.

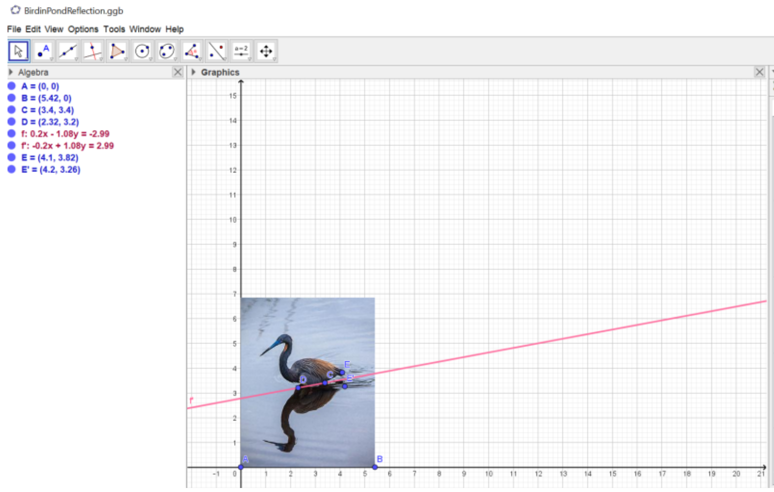

Figure 9. GeoGebra File Showing Photo and Line of Reflection

Reflections can often show up when taking photos of water, glass, or any other type of reflective surface. The photo above in Figure 9 shows a photo of a **reflection** of a bird in water with a **line of reflection** drawn in GeoGebra.

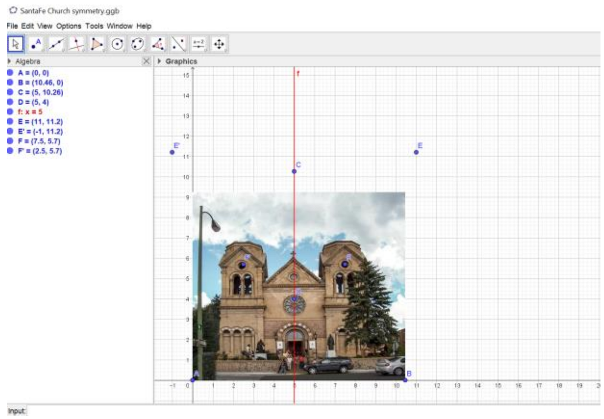

Figure 10. GeoGebra File of Symmetry of Photo of a Building with Symmetry

Figure 10 shows a photo inserted into the GeoGebra software, a line was drawn through the center of the photo and then a **point** was selected and **reflected along the line** to show them as **symmetrical** to each other.

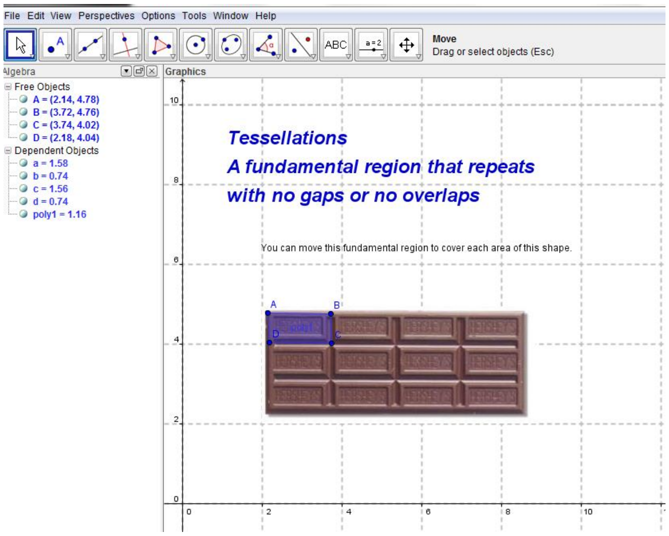

Figure 11. Photos of Tessellations-Using a Photo of a Chocolate Bar

Tessellating patterns are patterns that repeat with the same **fundamental region** covering a space, with no **gaps** and no **overlaps** like seen in the chocolate bar rectangular pieces seen in Figure 11. Students can use the GeoGebra and move the pieces to test their hypothesis of the same shape through repetition. By using GeoGebra and inserting photos into the software, students can then use the tools in GeoGebra to do the math, learn the vocabulary, and understand mathematical concepts better.

In Conclusion

Today young people intrigued by technology will construct and investigate geometric shapes and many math ideas with GeoGebra and will start enjoying math and even see some connections in mathematics and maybe even develop a passion for photography when teachers have them import photos into the GeoGebra software! It is important to make such connections when teaching math using GeoGebra and Photography because they:

- Show a purpose for math
- Better reach all types of learners even ESE and ESOL
- Make connections and relationships between math and photography while covering math concepts
- Employ emerging technologies in math with the real world
- Show practical applications to math in everyday life
- Employ innovative teaching in the classroom
- Stimulate excitement through Photography/Modeling
- May help students develop a passion for photography in the process

In a globally competitive world where it is more important to prepare our students for science, technology, engineering, and mathematics (STEM) fields, using GeoGebra software covers all the STEM areas. It applies math, science, and engineering ideas while using and applying the technology of GeoGebra along with digital photography to teach math and motivate and help develop a passion toward photography in our young people. What better way to learn math than to use pictures and photos and to see the math in them, applying the GeoGebra math tools to understand and apply the mathematics. Pictures still speak the most universal language, a picture is worth a thousand words as the saying goes, photographs can be so inspiring and kind! Our young people can become artists with their photography and also use the technology to understand the mathematics they have to learn in school! When our young people grow up they can still remain artists with a passion for photography! The GeoGebra technology, photography, and emphasizing math vocabulary and concepts when teaching can help teachers reach all types of students, especially our ESOL populations.

The suggested means to teach math to all students are by no means exhaustive. With the ESE learners, the use of the multi-modal approach that incorporates the multiple intelligence caters to students' short attention span as they are not expected to only sit still to learn the materials, going out and taking photos and then uploading and inserting them into GeoGebra and exploring all the math on them can be more exciting for young learners. With the English Language Learners (ELL), we have included the SDAIE (Specially Designed Academic Instruction in English) approach which supports the teaching of language acquisition while teaching the content area such as Math. Language objectives are incorporated in teaching Math concepts, therefore, ELL are given the opportunity to simultaneously develop their English language skills as they are learning Math. Math teachers today must work hard to eliminate and prevent any math anxiety their students may develop or carry with them (Furner & Duffy, 2002). Our children today are not only competing for jobs with others in the U.S.A., but with others from around the globe--being confident in their ability to do mathematics in this competitively global society. With these approaches and strategies mentioned in this article, content area teachers can also play a crucial role in the language learning process of their students and the primary teachers no longer have to be the sole responsible party in teaching ELL the English language; rather, both the content area teachers and the primary teachers can work as a team to help the ELL in their academic work. Each day the diversity of students grows within the confinement of our classrooms, therefore, teachers have to tirelessly keep abreast with their research of diverse teaching strategies to reach all students to learn. Teachers must see to it that they reach all students equally. Equity in mathematics instruction requires teachers to provide accommodations so everyone in the class can learn mathematics. The 'best practices" mentioned here can better assist any teacher in reaching all students mathematically while using technology like GeoGebra and photography which can really appeal to the learners and make math not only appealing but getting young people ready for a STEM world.

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Manuscript 1032

Strategy to Estimate Size

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Strategy to Estimate Size

by

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Introduction

Have you ever wondered exactly how much land and space is included in a national park?
In this paper, we will explore the vastness of a fictitious Park and calculate its total area.

We will:
1. Create a fictitious Park, determine a scale that can be used to perform calculations using a practical unit of measurement.
2. Develop a strategy that students can use to find the area of the Park.
3. Demonstrate the use of the strategy by solving for the area of the fictitious park.

We start with a real National Park

We investigated several National Parks in the United States (Acadia National Park, Yellowstone, Grand Canyon, Yosemite, Glacier, Denali, and Big Bend). Five out of the seven parks had irregular shapes that roughly resemble a circle. Using the scale provided on the image, we can estimate the size of the national park.

Figure 1. Map of Yellowstone National Park.

Referencing the scale of the map, a length slightly greater than 0.5 in. represents 10 miles. Placing the map on a grid allows us to see how many square-units occupy the region enclosed by the boundary.

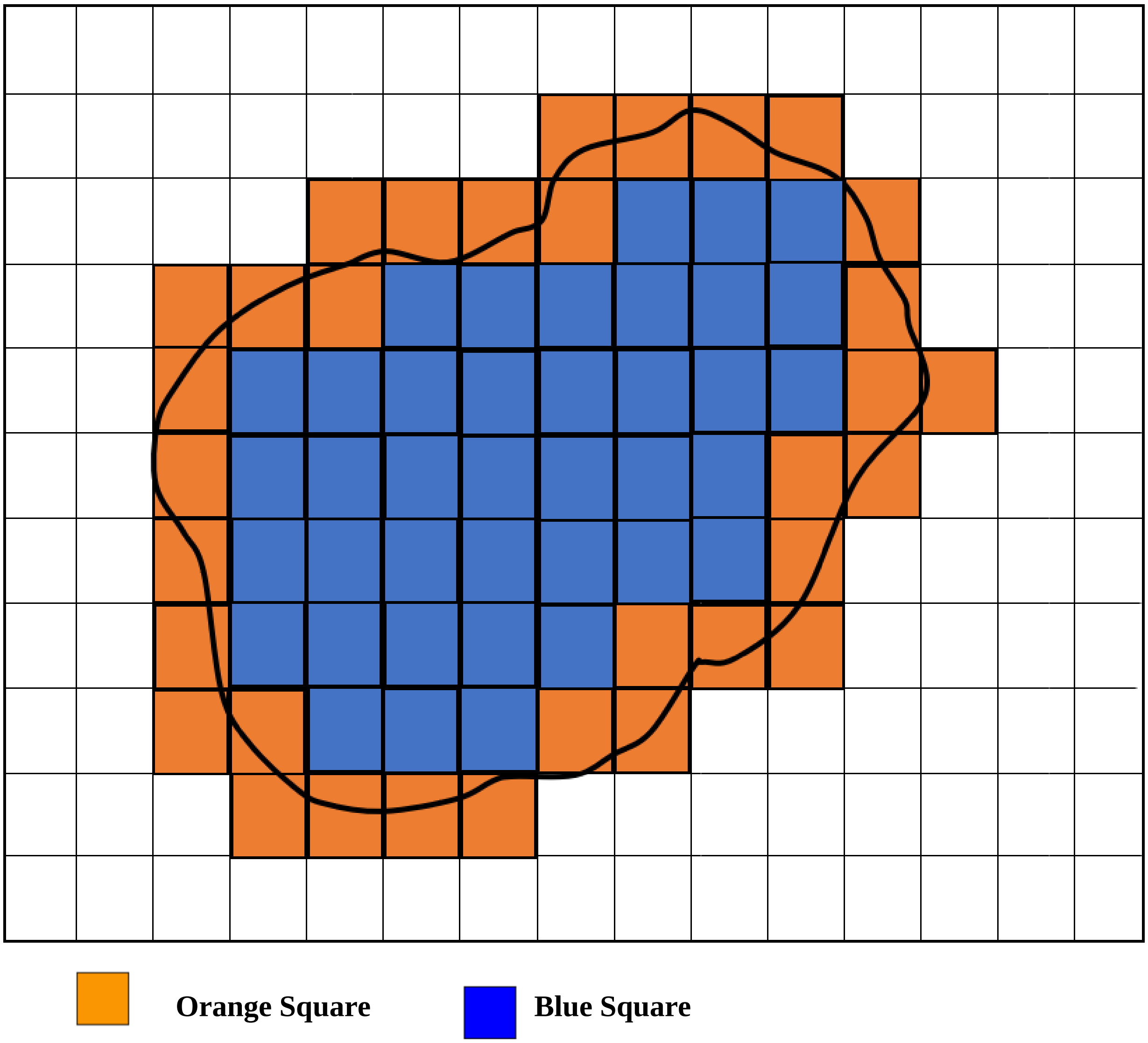

Figure 2. Outline of Yellowstone National Park (Depiction of Inner and Outer Area).

The edge length of each square is 0.44 inches which results in an area of $(0.44\ in)^2 = 0.194\ in^2$. for each square. Applying dimensional analysis (conversion factors) will allow us to find the area of the park in units of square miles. Before doing so, we need to calculate the amount of square units occupied by the figure. The inner area of the figure consists of the total number of squares completely contained within the contour, i.e., the blue squares…

$$A_{inner} = 39\ unit^2.$$

The outer area consists of the total number of orange *and* blue squares:

$$A_{outer} = 39 + 72 = 111\, unit^2.$$

Finally, the mean (average) of these values is an approximation of the area (Pirnot, 2017):

$$A \approx (A_{inner} + A_{outer})/2$$

$$A \approx 55.5\, unit^2.$$

This process will be explained in depth in the following sections. In the meantime, let us proceed with the estimation. The area of each square unit can be applied as a conversion factor, along with the scale provided on the map. The area of Yellowstone National Park is given by…

$$55.5\, unit^2 \times \frac{0.194\, in^2}{1\, unit^2} \times \frac{100\, mi^2}{0.25\, in^2} = 4{,}307\, mi^2.$$

Note that the actual area of the national park is 3471 mi^2.Using the equation for percent error allows us to compare the estimated value with the actual size of Yellowstone:

$$\%\, Error = \frac{|estimated\ -\ actual|}{actual} \times 100\%$$

$$= \frac{|4307\ -\ 3471|}{3471} \times 100\%$$

$$= 24.1\ \%\,.$$

Note how limitations in the measuring device will skew the results. For instance, suppose the ruler had a greater capacity to display significant figures. This would result in more certainty up to a greater number of decimal places. Unfortunately, however, 0.53 inches may be more accurate for the length of the scale (there could be any number following the digit "5"). If we report our measurements with a greater number of significant figures, perhaps we will obtain a more accurate estimation. Suppose the length of the line is 0.53 inches.

Let the scale of the map read 0.53 inches in the image to every 10 miles in the actual park. Therefore,

$$(0.53\ in)^2 \ = \ (10\ mi)^2$$

$$0.2809\ in^2 \ = \ 100\ mi^2\ .$$

Combining the above relationship into a fraction allows us to utilize it as a conversion factor:

$$55.5\ unit^2 \ \times \frac{0.194\ in^2}{1\ unit^2} \times \frac{100\ mi^2}{0.2809\ in^2} = 3{,}833\ mi^2.$$

Notice how this numerical value is much closer to that of the actual area. Using the scale of the map and conversion factors is a great problem-solving strategy.

Diagram (Map) of Our Park

Suppose a park takes the (irregular) shape of an arbitrary closed curve. Since the shape is not a polygon or a recognizable figure, there is no formula for perimeter or area that quantifies its size. The goal is to devise a method, allowing one to estimate the size of this park.

Method 1. Place the Figure on a Grid

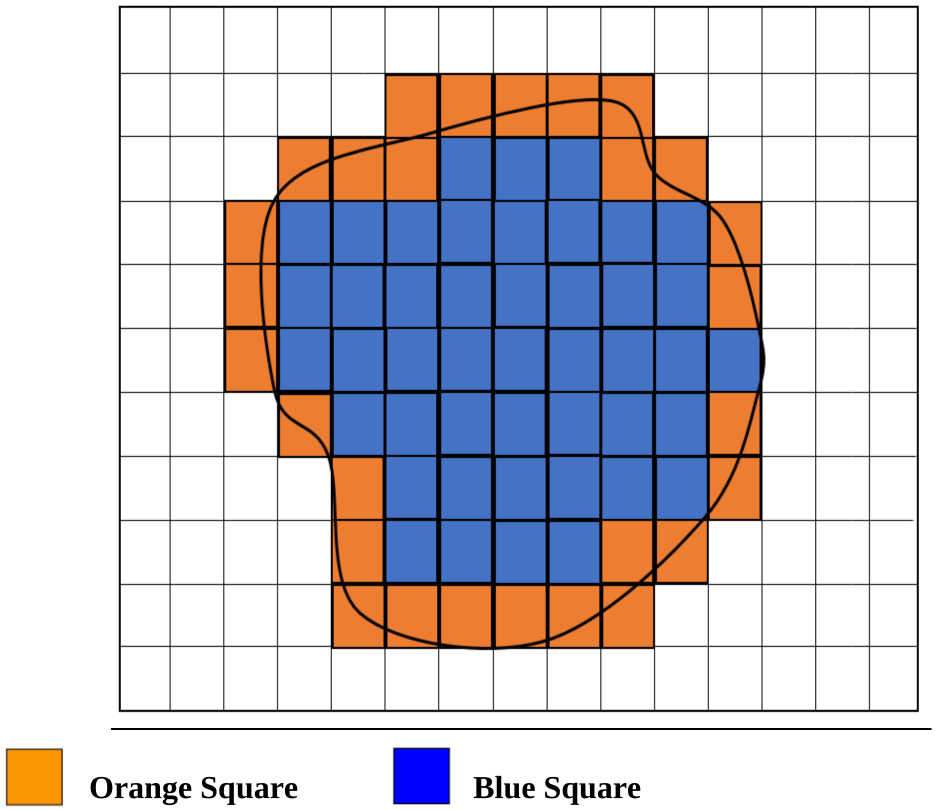

Figure 3. Outer and inner area of our national park.

Algebraic Expression to Estimate Size

All squares completely contained inside the shaded region represent the **inner area**. The orange squares are not fully enclosed by the boundary; rather, the contour crosses *through* these squares (this denotes the **outer area** of the figure) (Pirnot, 2017).

The area of an individual square is given by l^2, where l denotes the edge length of the square. The inner area is the sum of individual areas enclosed within the boundary. Since multiplication is repeated addition…

$$A_{inner} = n_{blue}\, l^2 \,.$$

The outer area is the sum of the orange squares *and* blue squares, collectively. It is very important to emphasize this to students. A common misconception is that *only* the orange squares constitute the outer area. The name is somewhat unfortunate because, as we mentioned, the outer area is not simply the orange squares; rather, it is the total area (squares) of the figure.

$$A_{outer} = n_{blue}\, l^2 + n_{orange}\, l^2.$$

The inner area is too small to represent the size of the actual park, whereas the outer area overcompensates. This suggests that the actual area must be some value in between. In order to approximate the area of the entire park, we will take the average of these two values (Pirnot, 2017):

$$A \approx \frac{1}{2}\,(A_{inner} + A_{outer})$$

Substituting in the expressions for the outer and inner areas:

$$A \approx \frac{1}{2}\,(\, n_b\, l^2 + n_b\, l^2 + \, n_o\, l^2)$$

Note that l^2 appears in all terms; therefore, we can distribute this factor outside the parentheses…

$$A \approx \frac{1}{2}\, l^2\,(n_b + n_b + n_o\,)$$

Combining like-terms gives…

$$A \approx \frac{1}{2}\, l^2\,(2n_b + n_o)$$

An equation has been derived which yields the approximate area of the park. Suppose one unit on the grid represents one decameter (1 Dm). Substituting this length into the equation allows us to find a numerical value for area:

$$A \approx \frac{1}{2}\, l^2\,(2n_b\ + n_o)$$

$$= (1/2)(1)^2\,[\,(2)(45) + 28]$$

$$= (1/2)(118)$$

$$= 59.$$

Therefore, the area of the park is approximately equal to $59\ Dm^2$.

Notice how it may be an inconvenience to individually count the number of squares. We can generalize our method to regions consisting of any number of square units, and those units can be calculated via $A = lw$.Suppose the presented problem asked you to find the area of an irregularly shaped figure. Again, if the area consisted of a finer grid (more squares), you would encounter yet *another* problem: what is the most efficient method to find the inner and outer areas? How can I find the number of blue squares (inner area), for example, without having to manually count them one-by-one? The following example illustrates how to solve this problem. Instead of using more squares, however, we will keep the example simple and use our current grid from *Figure 3.*

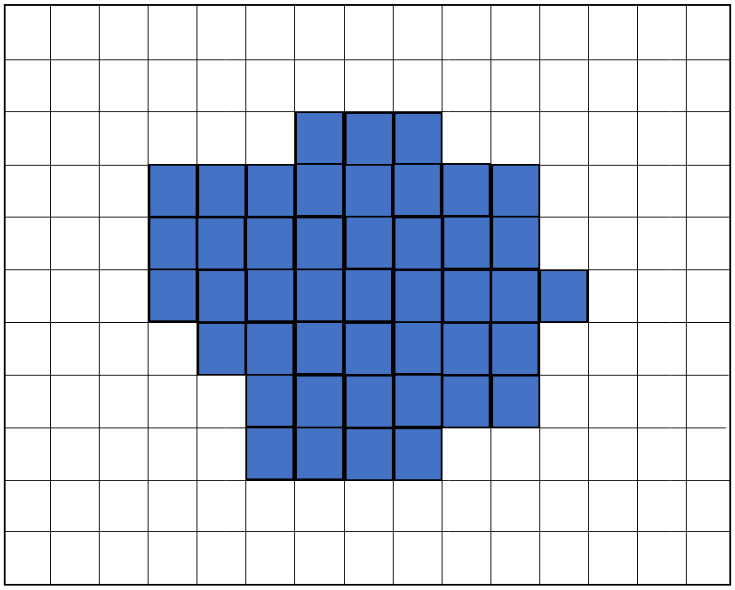

Figure 4. Depiction of the inner area of the park.

Placing additional square units in the unoccupied cells allows us to create a rectangle. As shown in the figure below, this allows us to calculate the area using its length and its width. Turning this problem into a previously encountered situation (finding the area of a rectangle) is a wonderful problem-solving strategy. These are strategies that should be highlighted as the instructor transitions from step-to-step sequentially, rather than being emphasized in an isolated context (when a problem is not being solved real-time). Moreover, emphasizing such methods is a great addition to a mathematics curriculum: making diagrams, organizing information, reading the problem multiple times, looking for patterns, and so on (Lawson, 1990). Developing such strong habits is essential for a student's problem-solving skills, as they are universally applicable to all problems.

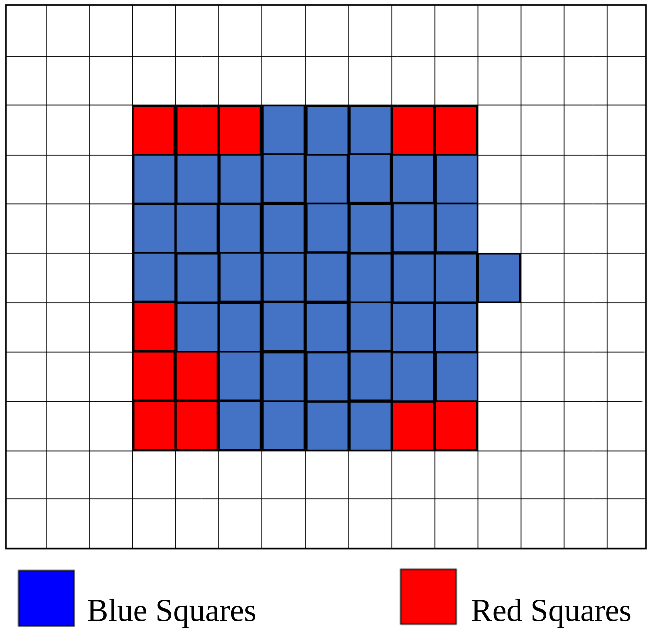

Figure 5. Inner area with additional areas $A_1, A_2, A_3, and\ A_4$.

Notice how another dilemma is encountered. There is a blue square occupying a column of the grid by itself. It is located four units left and six units up from the bottom right cell. We could occupy that column with red squares; however, there are more efficient ways to take this into account. In this case, let's find the area of this rectangle (as if the blue square was not there) and simply add the area of the blue square *after*.

$$A_{rectangle} = lw$$

$$= (8)(7) + 1$$

$$= 57\ units^2\ .$$

Finally, subtracting away the additional (red) areas will leave the actual inner area (the total number of blue squares). Letting A_1denote the area of the (red) upper left corner, A_2denote the area of the upper right corner, and so on...

$$A_{inner} = A_{rectangle} - (A_1 + A_2 + A_3 + A_4)$$

$$A_{inner} = 57 - (3 + 2 + 5 + 2)$$

$$A_{inner} = 45\, units^2\, .$$

Method 2. Approximating the Area with a Circle

Although the shape is highly irregular, we can estimate the size of the park using a circle. Using this alternative method ensures the previous approximation was valid, if we obtain a consistent numerical value. Utilizing a shape that is more familiar to us is a wonderful problem-solving strategy, because it simplifies the given task. Employing Polya's Four-Step method, it is clear that we are identifying the main problem: how can we devise a way to quantify the size of an irregularly shaped closed curve? Is there a shape that this figure most closely resembles? If so, what formula can we apply?

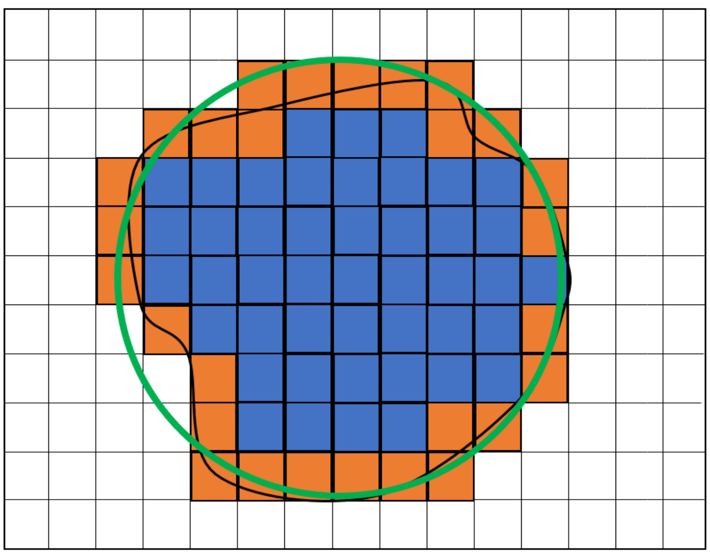

Figure 6 . Estimation of the area using a circle of radius 4 units.

Applying step two of Polya's Four-Step method, we need to devise a plan. Using *Figure 1*, we constructed a circle with a radius of roughly four units (*Figure 6*). The area of the circle is given by $A = \pi R^2$. Plugging in the value for radius R allows us to determine a numerical value for the area. Employing step three of Polya's problem-solving method…

$$A = (3.14)(4)^2 = 50.24\, Dm^2.$$

The units of area should be units of length squared. The radius is measured in decameters, so it logically follows that our area will be units of decameters squared.

There is one last way to quantify the size of this national park. Although the area is a great representation of its size, the perimeter of the park has not been calculated. Geometrically, the perimeter of a circle is its circumference…

$$C = 2\pi R$$

$$25.12\ Dm.$$

Problem-Solving Model Utilized: Polya's Four-Step Method

Step 1: Understand the problem

Identify the problem. What are we asked to find? What is the fundamental principle? Is there a particular method or equation that applies to this specific problem?

Step 2: Devise a plan Assign variables to given quantities, set up algebraic equations, construct diagrams, and consider formulas which may apply to the given situation. Organizing your data is a great starting point when approaching a challenging problem.

Step 3: Execute the plan

Isolate any desired quantities and make inferences from any diagrams. Although some diagrams are provided by the author of this particular problem, it may be necessary to expand on the given schematic. Check that the final numerical result has the appropriate units and sign. For instance, suppose you are computing the volume of a cylinder. A negative volume is not possible; furthermore, the volume should be expressed in units of length cubed.

<u>Step 4: Looking Back</u>

<u>Students have done this when they used the circle in addition to using the grid shapes to find the area. Answers the question: Is there another way to find the solution? Is there a more elegant solution? It is more than just verifying calculations are correct. (p. 14 -15 of Polya's *How to Solve it.*</u>

Conclusion:

Using Polya's Four Step Method, this problem was approached in a similar way that most students would attempt to solve this problem. Relating new information to previous information or applying previous skills to new types of problems are the true measure of the problem-solving skills teachers strive for their students to possess. The formulas that were used to approximate the area of our National Park started relatively simple with the area of a rectangle and a circle. This type of artifact can have classroom applications from middle school to post-secondary education because it depends on simple formulas that students are introduced to very early. All of the results of our areas were relatively consistent which supports our approximations being correct. All calculations have been included and this could easily be recreated in any classroom and serve as a guide; as well as being added to because even though we included several ways to solve the same problem, the beauty of problem solving is that there can always be another way.

Use in the Classroom

The main goal of this project is for students to use whatever knowledge they have to estimate the size of a National Park. In a fifth-grade classroom this project could be modified to test mastery of multiple common core standards. The standards are CCSS.MATH.CONTENT.5.G.A.1 and CCSS.MATH.CONTENT.5.MD.A.1. These standards address student ability to plot points on a coordinate plane and convert measurements within the same measurement system. By adding an aspect to the problem where students have to not only estimate the size of a park, but also define the ecosystem in the park, allows students to show mastery of the Next Generation Science standard 5-LS2-1. This activity blends together the benefits of project based learning and interdisciplinary instruction. The goal of interdisciplinary instruction as well as Project Based Learning are used to prepare students for real world problem solving, by working with a group of students to discover something.

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Transformations

Manuscript 1033

Creative Geometry Games

Hui Fang Huang Su

Dylan Mandolini

Bhagi Phuel

Creative Geometry Games

by

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Introduction

The geometry game presented in this article was inspired by Bright and Harvey's (1988) *Learning and fun with geometry games*. In their article, Bright and Harvey (1988) propose three interactive games: Polyhedron Rummy, Polygon Rummy, and That's Stretching it. Polygon Rummy is a game pertaining to plane geometry where the instructional objective is to construct a figure using lines and angles, whereas the goal of polyhedron rummy is to construct a solid using faces and angles. Although Bright and Harvey focused on constructing shapes, they also suggested presenting pre-constructed shapes to students, allowing them to classify and identify them (Bright & Harvey, 1988). Our activity does just this.

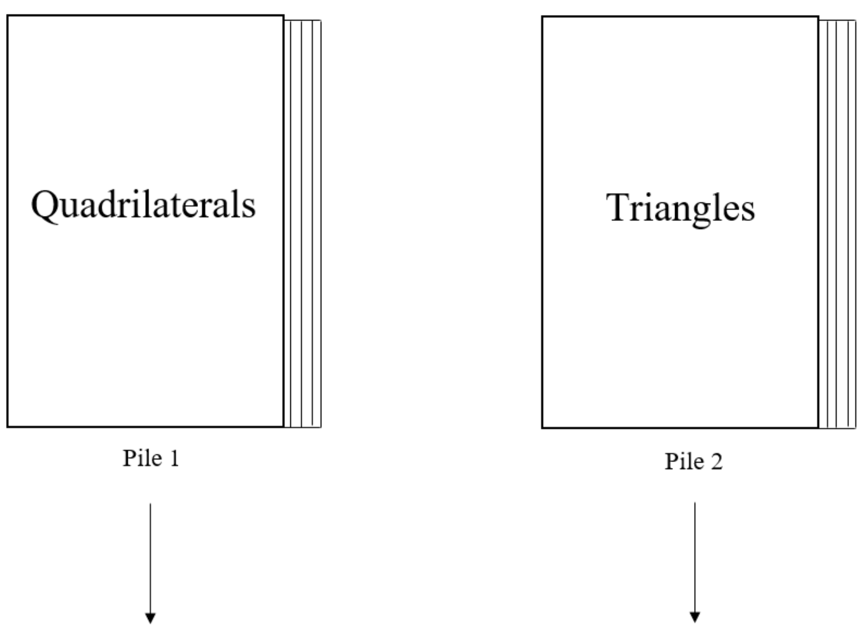
Research-Based Geometry Game

The geometry game consists of multiple piles of cards. Pile 1 is a deck consisting of cards pertaining to quadrilaterals. Each student draws five cards from this pile (unless the students opt for pile 2). Upon doing so, students may obtain many different classifications of quadrilaterals, such as trapezoids and parallelograms. Pile 2 consists of triangles. If a student chooses to draw five cards from the deck of triangles, they may receive isosceles triangles, equilateral triangles, right triangles, and so on.

Suppose a student's hand of cards is obtained from pile 1. Adjacent to pile 1 is another stack of cards which we will call pile 1.1. This stack consists of the different classifications of quadrilaterals. A student immediately draws a card from pile 1.1 subsequent to drawing form pile 1. If the card reads “trapezoid”, for example, then the student is to differentiate which of their five quadrilaterals is a trapezoid. If the student correctly identifies which of their quadrilaterals is a trapezoid, then they earn two points and move onto tier 2. Moreover, suppose the student’s hand does not contain any trapezoids. In this case, the student can declare that “zero of my cards have the indicated figure”. This student will still earn two points; however, they will not proceed to tier 2. Note that pile 2.1 will denote the classifications of triangles, e.g., right, equilateral, isosceles, and so on.

Tier 2 consists of rolling a six-sided die. The die is labeled one through six on each of its faces. Once a student has successfully identified the category of their figure(s) (and has earned two points), they must analyze their card(s). Continuing with the example of a trapezoid, suppose our student has two trapezoid cards in their hand. This hypothetical student may choose which card they would like to work with. The student will select one trapezoid card and discard the other back into the deck. The trapezoid card has various numbers at each side of the trapezoid, at each angle, etc. These numbers can range from one through six (just like the die). Consider the case where this student rolls a “2” on the die. As a result, the student is to reference their card and calculate the corresponding quantity designated by “2”. This can be an angle, a length of a side, or a command, such as “show an example of a rotation of the figure". If the student correctly answers the question or command signified by the designated number, they earn four points.

Finally, not all cards have number designations up to six. For instance, a single card may have a 1, 2, and 3 (each designating the angle measure at each vertex of a trapezoid, for example). What happens if a student rolls a 4, 5, or 6 on the die? In this case, the student must address a true/false question in a third pile of cards, labeled pile 3. The true\false question pertains to any topic in geometry such as tangent lines, fractals, rotations, translations, etc.



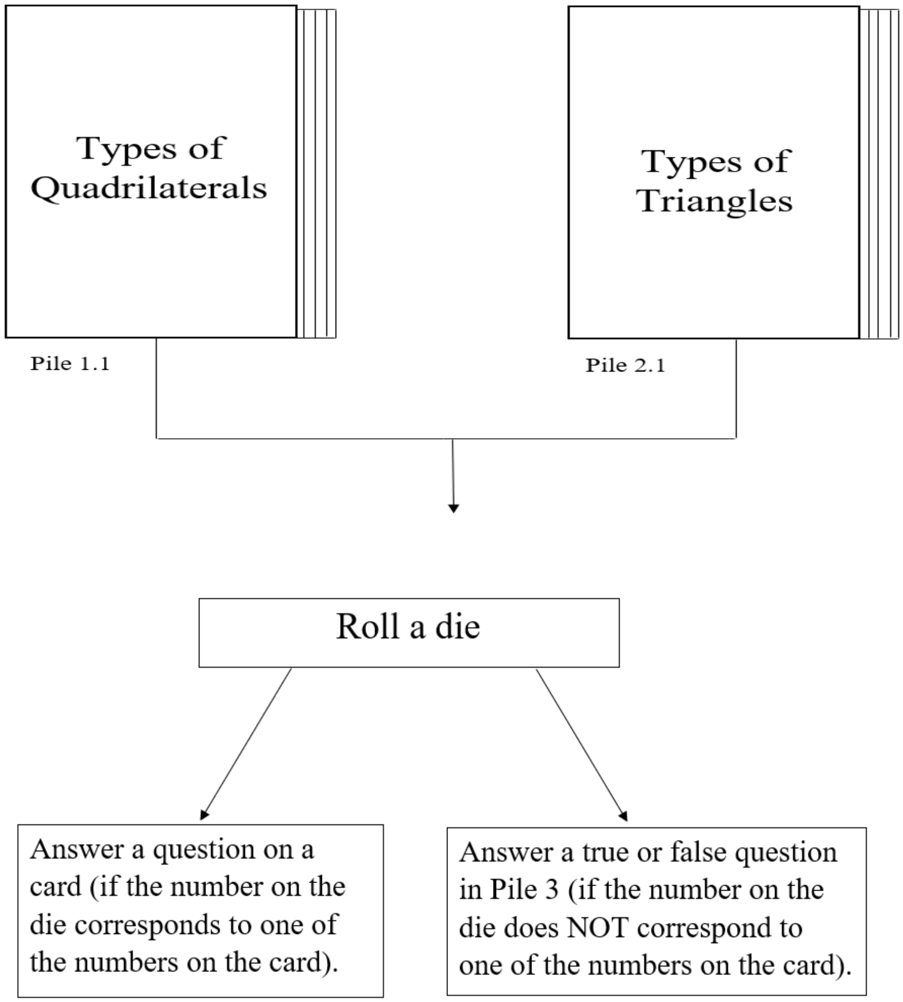


Fig. 1. Schematic of how the game proceeds

In the deck of cards, there will be many examples of the same shapes and diagrams. As a matter of fact, some problems will be exactly the same, aside from the numbers themselves for the lengths and the angle measures. This ensures repetition to reinforce the concept. According to Bright and Harvey (1988), “one of the strongest features of game playing is that a game presents students with many very similar problems that are therefore solved using the same problem-solving techniques” (p. 22). The figure below represents a hand of cards drawn:

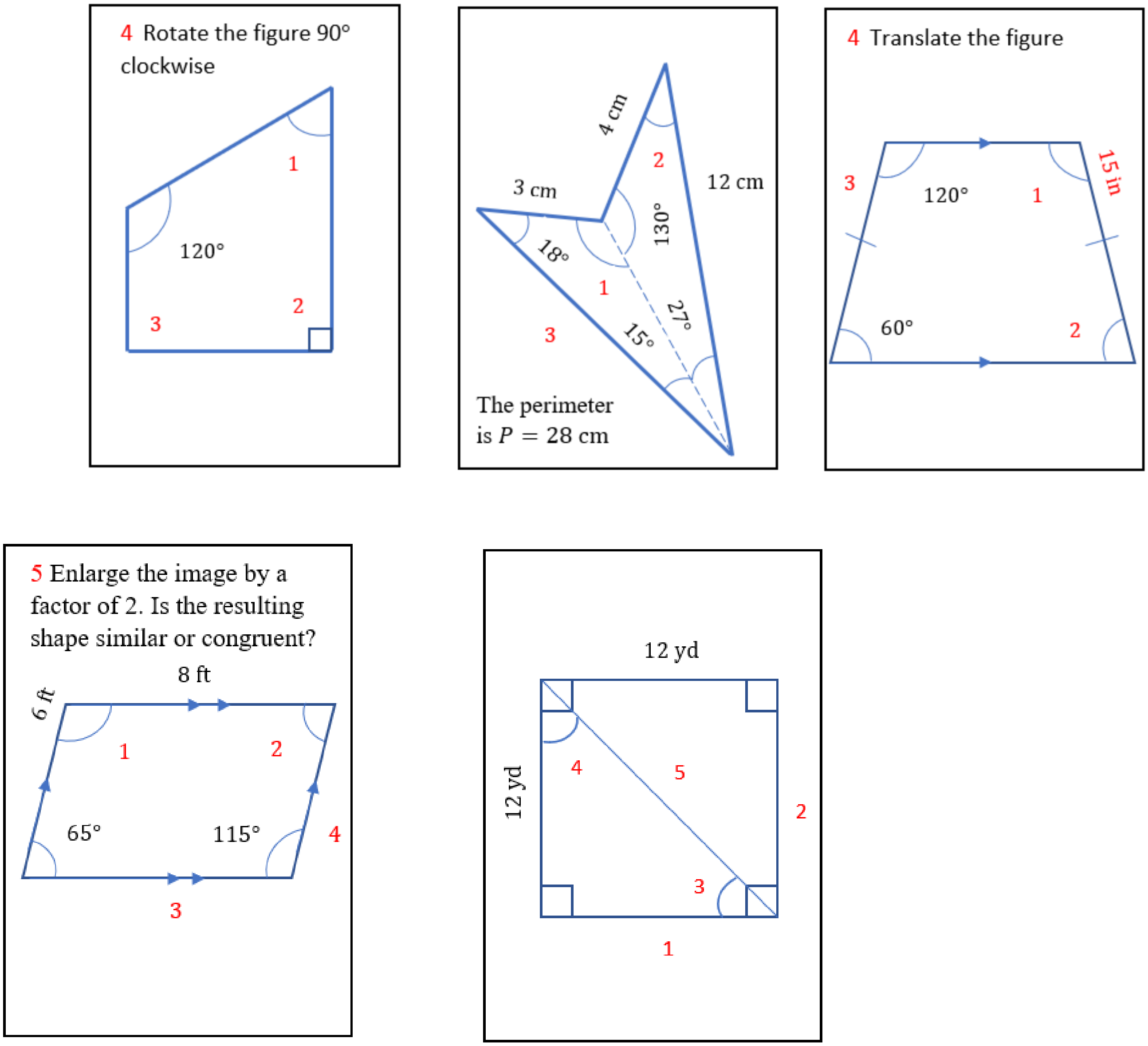


Fig. 2. A hand of five cards drawn by Pile 1 (quadrilateral pile)

Notice that some cards vary in difficulty. As a matter of fact, some may require the applications of theorems. Most of these cards are based off the generalization that, for any quadrilateral, the sum of interior angles is equal to 360°. For instance, the second card shows a quadrilateral represented as two triangles. The sum of interior angles of each triangle is equal to 180°. “Geometry games can accomplish a number of desirable instructional goals. For instance, they can furnish settings in which students visualize or construct geometric figures, see specific examples of general results or theorems, and apply logical reasoning skills in informal situations" (Bright & Harvey, 1988, p. 22).

What is the purpose behind Pile 3? Pile 3 consists of true-false and free-response questions. This allows students to participate in communicating with mathematical language. According to Bright and Harvey (1988) in *Games, geometry, and teaching*, the classification of quadrilaterals is usually taught through definitions (which they call a comprehension-level activity) and through grouping shapes based on commonalities (an analysis-level activity). Pile 3 allows one to achieve the instructional objective of comprehension-level activities mentioned by Bright and Harvey. For instance, the definition of a quadrilateral will be read to students and they can determine whether the given statement is true or false. On the other hand, Pile 2 engages students in analysis-level activity, forcing them to analyze which polygons are classified as parallelograms, for example. In this case, each of these polygons would be quadrilaterals consisting of two pairs of congruent (and parallel) sides. The following image depicts examples of cards in Pile 3 that use comprehension-level activity (since they are focused on defining the essential features) (Bright & Harvey, 1988). As a matter of fact, some show analysis-level activity as well.

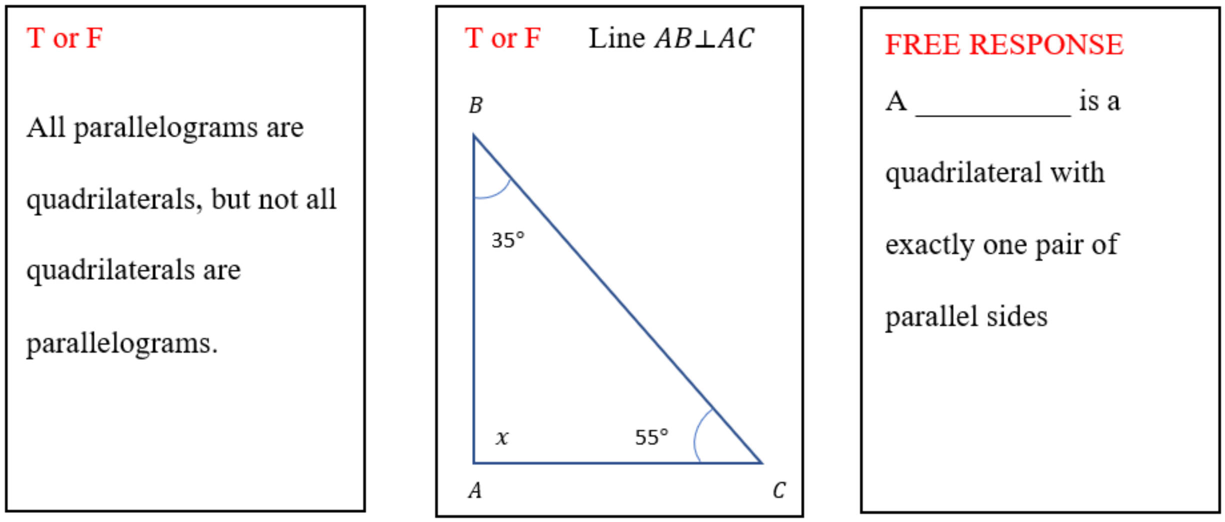


Fig. 3. Examples of cards that constitute Pile 3.

Notice that cards 1 and 3 focus on comprehension-level, whereas the second card actually focuses on the analysis-level. This is because the student has to make a logical leap, deducing that all right angles are congruent and have an angle measure of 90°.

Thus, $AB \perp AC$ iff $x = 90^\circ$.

Since the sum of interior angles in a triangle must be equal to 180°:

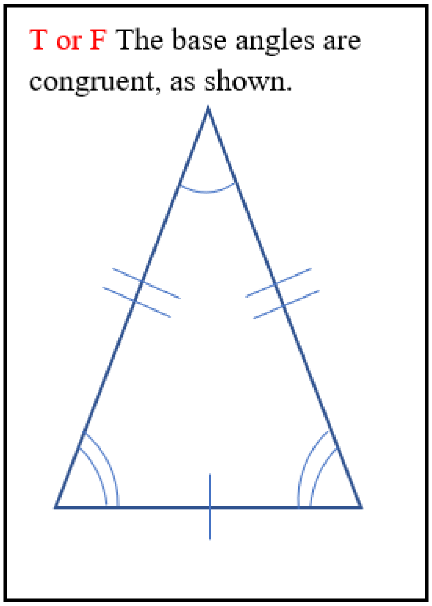
$$x + 35^\circ + 55^\circ = 180^\circ$$

$$x + 90^\circ = 180^\circ$$

$$x = 90^\circ.$$

$$\therefore\ AB \perp AC.$$

The next two examples of cards are also wonderful ways to illustrate the properties of parallelism and perpendicularity to students:



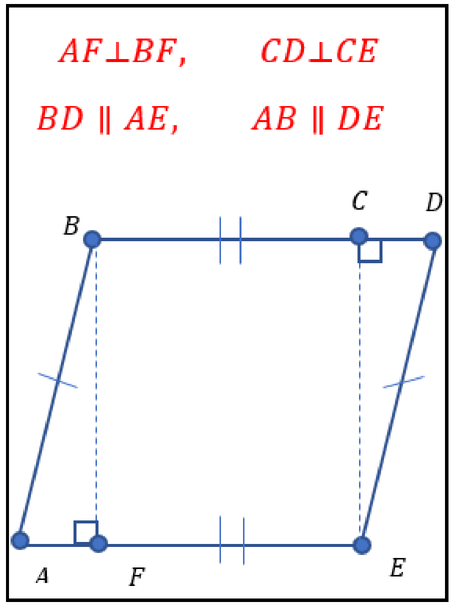


Fig. 4. Examples of an isosceles triangle and parallelogram.

In addition, the game can be extended even further by integrating technology. A study implemented by Turk and Akyuz (2016) indicated that the implementation of dynamic geometry systems (DGS) had a positive impact on student achievement and attitude in an eighth-grade geometry course. Students emphasized how the ability to draw and drag figures assisted them with comprehension (Turk & Akyuz, 2016). Since students commented on the interactive nature of DGS, the activity proposed in this paper can be extended via GeoGebra. In particular, one of the instructional objectives of this game is to teach transformations of geometric figures; hence, a dynamic software is ideal for such a lesson plan.

Furthermore, Turk and Akyuz (2016) prescribed a geometry achievement test (GAT) and geometry attitude scale (GAS) following the dynamic computer-based lesson. The results showed that students had an improved attitude towards geometry when they studied triangles using DGS. This reinforced the desire for the implementation of GeoGebra in this paper.

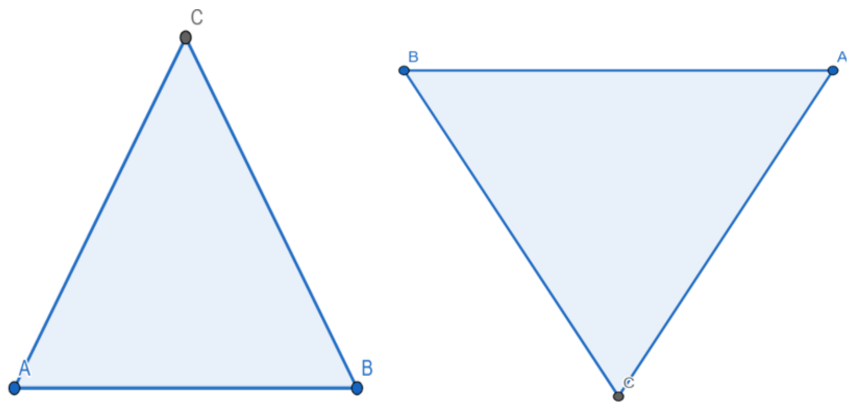


Fig. 5. The rotation of a triangle by 180 degrees using GeoGebra.

This allows students to easily demonstrate transformations on the computer rather than a dry erase board. This is also great for enlarging or reducing the size of figures.

Turk and Akyuz (2016) asserted that students even described the lesson as fun and meaningful when they manipulated triangles on GeoGebra to investigate the relationships between sides and angles. "Some of the students of the treatment group stated that their learning was easier and more meaningful than their past learning experiences since they explored topics themselves instead of memorizing formulas and theorems" (Turk & Akyuz, 2016, p. 101).

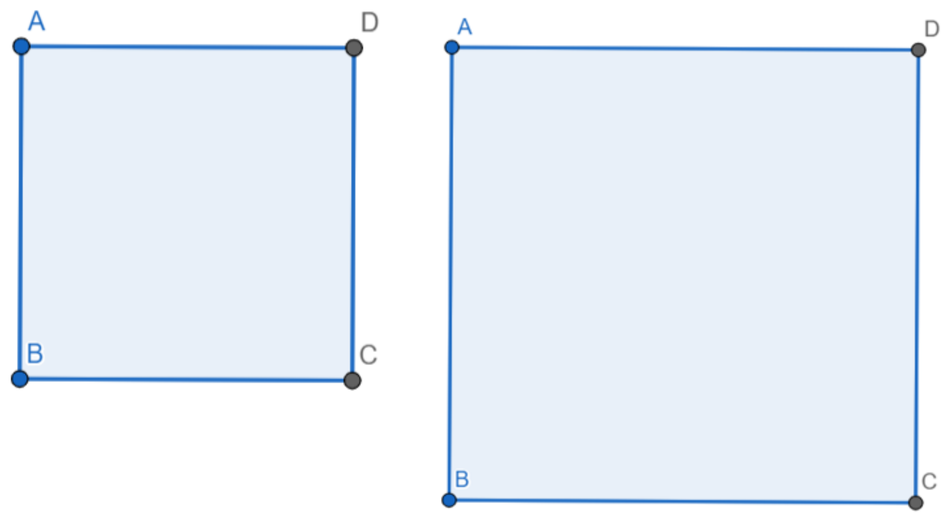


Fig. 6. Enlarging a square by some factor via GeoGebra.

Summary of Student-Teacher Experience and Evaluation

According to "Ideas for the Classroom" (2006), there was a mathematician named Gill Hatch (1937-2005) who was known for creating exciting algebra and geometry games for students. One of Hatch's works was a popular booklet called Jump To It! In this booklet, Hatch devises a series of games intended for students in the age group ranging from nine to 14 ("Ideas for the Classroom", 2006). "As Gill's work developed, Gill made it clear that she wanted to devise investigations and activities that were rooted in the curriculum and that developed the necessary techniques and skills alongside understanding and the ability to think mathematically" ("Ideas for the Classroom, 2006, p. 6).

Aside from the booklet for ages nine through 14, Hatch also developed an interesting geometry activity for older students. The game consisted of assigning a cylinder with a given volume to a group of students. Using the volume of the cylinder, students were asked to devise a creative way to determine the minimum and maximum surface area of that cylinder (“Ideas for the Classroom”, 2006). According to “Ideas for the classroom” (2006), students had the experience of finding the minimum and maximum surface areas via *Excel* or via the application of calculus.

After observing students playing games for years, Gill Hatch asserts that the teacher must be actively present throughout the geometry game (“Ideas for the Classroom”, 2006). When games are implemented as a pedagogic device, a teacher must have a deeper analysis of students’ behavior in order to observe interesting outcomes, facilitate the game, and organize the game to where preparation does not consume class time (“Ideas for the Classroom”, 2006). Hatch observed the following outcomes while incorporating geometry games in class: students received additional practice, pupils had a context for mathematical reasoning, individuals were welcomed to make generalizations, students were introduced to challenging ideas and situations, and children were supported by teachers and peers to work above their normal level (“Ideas for the Classroom”, 2006).

Moreover, Hatch devised a highly interactive game called I-Like: Quadrilaterals. The strength behind this geometry game is that students work as one whole group alongside the teacher. The teacher shows a collection of quadrilaterals to the class (labeled one through ten, for example). With a specific property in mind, the instructor states “I like quadrilateral number six”, for instance. Based on this statement, students ask the teacher “do you like quadrilateral number nine?” Based on the teacher’s response, the students attempt to predict the property of the quadrilateral that the teacher has in mind (“Ideas for the Classroom”, 2006).

Bower (2004) provides a brief synopsis of Lynette Long’s book “Groovy Geometry: Games and Activities that Make Math Easy and Fun”. In addition to Bower’s synopsis, she describes her experiences with incorporating the book’s games and activities in the classroom. According to Bower (2004), her students were inclined to a particular game where the objective is to design a storybook. In “Shape Storybook”, students were expected to construct a polygon with more than four sides (Bower, 2004). As a matter of fact, the various games and activities can be adapted into a project as a follow-up activity (Bower, 2004). Bower (2004) describes her experience by commenting “my students enjoyed completing their project, and I enjoyed the students’ creativity” (p. 352).

Furthermore, another game in the book is called “How Tall”. This activity allowed Bower’s students to see applications of similar triangles. The objective of the game is to predict the height of an object using concepts of similarity (Bower, 2004). Similar to the previous game, this activity can also be adapted into the form of a project or incorporated into any geometry lesson. “Groovy Geometry includes valuable resource activities for hands-on discovery of geometric concepts. This book would be an excellent addition to any mathematics classroom" (Bower, 2006, p. 352). It is important to note, however, that each game has a section stating the required prerequisite material, such as terminology (Bower, 2004). Hence, when designing a geometry game, it is important to consider whether topics need to be reviewed before introducing them to students.

Conclusion and Suggestions

As stated by Bower (2004), many mathematics games can be applied in the classroom, adapted into follow-up activities, or even used as projects. It is important to incorporate these games in any form within a lesson plan. Furthermore, the synopsis of the articles suggests that games are successfully implemented when the teacher has a strong role as a participant. When designing a game, consider an activity similar to Gill Hatch's "I Like: Quadrilaterals", where the students must interact with the teacher, as proposed by "Ideas for the Classroom" (2006).

Similarly, not only do games provide excellent means for a teacher to participate, but they allow for the teacher to analyze their students. This provides the opportunity for the teacher to facilitate the game and observe ("Ideas for the Classroom", 2006). Additionally, designing a game has the practical purpose of implementing a point-system to quantify student-achievement. Bright and Harvey (1988) proposed that teacher-observation and point-systems during mathematics games serve as tools for informal assessment. Following the game, designing a corresponding project or follow-up activity will serve as a formal assessment. Therefore, it is suggested that a game has a point-system that accurately measures students' correct responses, teachers should have an active role in the game, and a formal assessment should still be provided (the game itself is not sufficient as a formal assessment; rather, an informal measure).

Finally, a game's strength is exponentiated when technology is incorporated. Similar to our paper *Fun with Measurements*, Gill Hatch used *Excel* in a game where students predicted the surface area of cylinders ("Ideas for the Classroom", 2006). Thus, when a game consists of measurements with many data values (such as cylinders of varying radii), *Excel* may be an excellent tool to teach students graphing data. Last but not least, another excellent tool to consider while designing a mathematics game is GeoGebra. Similar to Turk and Akyuz (2016), this paper demonstrates how GeoGebra is utilized to demonstrate transformations of plane figures. This allows students to visualize rotations, translations, and the scaling of geometric figures.

Although these recommendations are paramount to the design of a mathematics game, it is important to remember that these ideas can be extended to any mathematics course. It is strongly encouraged to incorporate similar games in algebra courses, for example. As a matter of fact, the game "I Like: Quadrilaterals" could be modified into a similar game called “I Like: Polynomials”, for instance. Perhaps the objective of this game would allow students to predict whether a teacher is describing a quadratic, cubic, or quartic function, etc. based off the properties of its graph. Extending this idea even further, students can graph the polynomial using *Excel* or GeoGebra and analyze its transformations, observing how the parent graph changes as the values of the leading coefficient is adjusted, and so on. Not only the possibilities are limitless, but the students will thoroughly enjoy themselves.

Lesson Plan (Include Technology to reinforce game)

Geometry Topic: Perpendicularity, parallelism, tangency, congruency, similarity, and transformations (reflections, translations, rotations, scaling, etc.) for exemplified with quadrilaterals and triangles.	**Grade Level & Course for Lesson:** The grade level for this geometry game pertains to grades eight through ten.
Learning Objective(s): The instructional objective is to classify various polygons as quadrilaterals. ● Students are expected to distinguish between parallelograms and trapezoids using concepts of parallelism. ● Similarly, using the concept of perpendicularity, students can differentiate between a rhombus and a square.	**Essential Question:** How can we classify and describe trapezoids, parallelograms, and triangles? What are the distinguishing features of a rhombus or a rectangle?

Content Standards (Focus):	**Standards for Mathematical Practice:**
1. *CCSS.MATH.CONTENT.HSG.CO.A.1* 2. CCSS.MATH.CONTENT.HSG.CO.A.2 3. CCSS.MATH.CONTENT.HSG.CO.A.3 4. CCSS.MATH.CONTENT.HSG.CO.C.10	1. Know the definition of perpendicular line and parallel line 2. Represent transformations in the plane. Compare transformations that preserve angle and distance versus those that do not. 3. Transformations of parallelograms, trapezoids, and regular polygons 4. Base angles of an isosceles triangle are congruent, the sum of interior angles in a triangle is equal to 180 degrees, etc.

Estimated Time Frame:

This game is based on an activity by Bright and Harvey (1988) called Polygon Rummy. Bright and Harvey suggested 20 minutes for students to construct polygons using various sides and angles. Since our activity involves classifying and performing calculations, we suggest a duration of 30-45 min per session. Furthermore, our deck consists of multiple piles and more materials are required, in general for this activity.

Prior Knowledge (Coherence):

- Students are expected to have a general understanding in complementary and supplementary angles, since this property is used to find missing angles inside various triangles and quadrilaterals.
- Students should have a strong foundation in the Pythagorean theorem, since it is utilized to calculate the length of a diagonal.
- Students should be familiar with vocabulary such as “base”, “leg”, and so on.

Assessment/Performance Task:

The **informal assessment** consists of teacher-observation during the geometry game session. Moreover, the quantification of points provides another form of assessment. The **formal assessment**, however, consists of a worksheet done in class and a follow-up activity.

Formative Assessment:

1. **Triangles and Quadrilaterals: Trapezoids and Parallelograms Worksheet:**

The purpose of this assessment is to measure students’ knowledge in perpendicularity and parallelism, extending those ideas to aid in the classification and distinction of various geometric shapes. Furthermore, students are exposed to translations, reflections, rotations, etc. The instructor will informally assess the students’ understanding using a point-system and teacher-observation.

- What classifies a polygon as a quadrilateral
- What classifies a quadrilateral as a trapezoid (describing the nature and number of its parallel sides)

- What classifies a quadrilateral as a parallelogram (describing its number of parallel sides)
- What is the altitude of a triangle? Is it always perpendicular to the base of the triangle?
- Do all right triangles contain two perpendicular line segments?
- Identify whether the quadrilateral or triangle experienced a rotation, reflection, etc. Upon transformations of two congruent figures, are they still said to be congruent? Are lengths and angles preserved? What if the image is enlarged by some factor? Are the angles still congruent?

2. **Discussion**

- Do parallel lines ever intersect?
- What shape is a parallelogram with 4 congruent sides and 4 right angles?
- What is the difference between a square and a rhombus?
- If a triangle contains two perpendicular legs, what type of triangle is this?

3. **Homework Assignment (Follow-up Activity)**

Please see the handout at the bottom.

Teacher preparation/Material List:

The following list is the set of required materials for this activity**:**

- Polygon playing cards
- 6-sided die (numbered 1 through 6)
- Dry erase boards for each student (to perform calculations and draw shapes)

Hands-on Manipulatives:	**Technology Tools/ Websites:**
Dry erase markers and small dry erase boards.	Computer and access to Geogebra website

Lesson Description:

1. The instructor will assign a student to be the card dealer. This geometry game has a deck of cards divided into many separate piles. As mentioned in the directions, the first pile of cards consists of questions pertaining to quadrilaterals, whereas the second pile pertains to triangles. The next two piles 1.1 and 2.1 consist of the different categories of quadrilaterals and triangles. Finally, one stack of cards entirely consists of true-false and free-response questions. **(5 MINUTES)**

2. Students will complete the warmup at the beginning of class prior to the PowerPoint presentation **(5 MINUTES)**.

3. After completion of the warmup activity, students will each receive 5 cards from the designated card dealer. Upon receiving their first five cards, they must draw from either 1.1 or

<table>
<tr><td colspan="3">2.1. Each student is given roughly 10 seconds to make a decision. This decision is regarding which cards in their hand fit the category specified by the card drawn in piles 1.1 or 2.1. For instance, if 1.1 reads “trapezoid”, this suggests that the students must determine which of the 5 cards in their hand has a trapezoid. For a class consisting of 25 students, for example, this would take about 4-5 minutes. (5 MINUTES)

4. The game is now fully in session. Students that correctly categorized the cards in hand will proceed to the next round. They draw a card that involves a calculation at this point, unless the die rolled results in a number that is not available on the card. If so, they must draw a card from the true-false or free-response pile. Since this step involves calculations, students may require more time on this portion of the game. (30 minutes)

5. The game is now completed. Student-achievement during the game is assessed, informally, via teacher-observation and the point-system. The last 15 minutes of class is dedicated to the formal assessment. A worksheet is administered to the students. (15 minutes)</td></tr>
<tr><td colspan="3">Teacher Notes:

Review the definition of a polygon before starting the game session. Additionally, discuss various transformations of geometric figures, such as reflections, rotations, translations, etc. Be sure to observe the class and actively participate. According to the research-based activities, it is important to analyze how your students respond to the game.</td></tr>
<tr><td colspan="3" align="center">Application:

Students will learn to describe features of plane figures using vocabulary such as “perpendicular” and “parallel”. These words are important for the classification and description of kites, parallelograms, and trapezoids.</td></tr>
<tr><td colspan="3" align="center">Differentiation/Accommodations</td></tr>
<tr><td>ESE:

Numerous accommodations will be made for ESE students:
Working with a partner
Practicing the game beforehand, that way the student is comfortable with following the instructions real-time</td><td>ESOL/WIDA:

The instructor can assist students in reading the questions displayed on the card, writing the question on the board, or phrasing the question in a different way.</td><td>Enrichment:

To ensure that students are challenged and engaged, the instructor can ask the student an EXTENSION of the question provided on the card. This is a hypothetical scenario that the card does not ask. For instance, after the student provides a correct response, the teacher can say “very good, would that statement also hold true for an equilateral triangle? Why do you think so?”</td></tr>
</table>

Formal Assessment (Worksheet)

1) **State true or false**

    a) **Each angle of rectangle is a right angle.**

    b) **The opposite sides of a trapezium are parallel**

    c) **The opposite sides of a rectangle are equal in length**

    d) **All the sides of a rhombus are of equal length**

    e) **All the sides of a parallelogram are of equal length.**

    f) **A triangle has three sides**

    g) **A triangle may have four vertices**

    h) **Every right triangle is scalene**

    i) **Each acute triangle is equilateral**

    j) **No isosceles triangle is obtuse**

Answer

a. **True**

b. **False**

c. **True**

d. **True**

e. **False**

f. **True**

g. **False**

h. **False**

i. **False,**

j. **True**

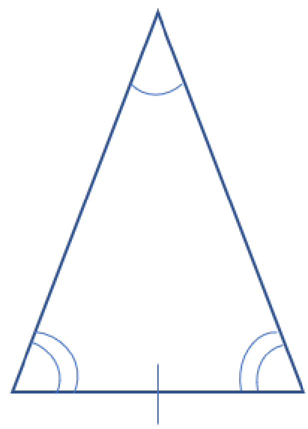
Follow-Up Activity

Polygons Worksheet: Quadrilaterals and Triangles

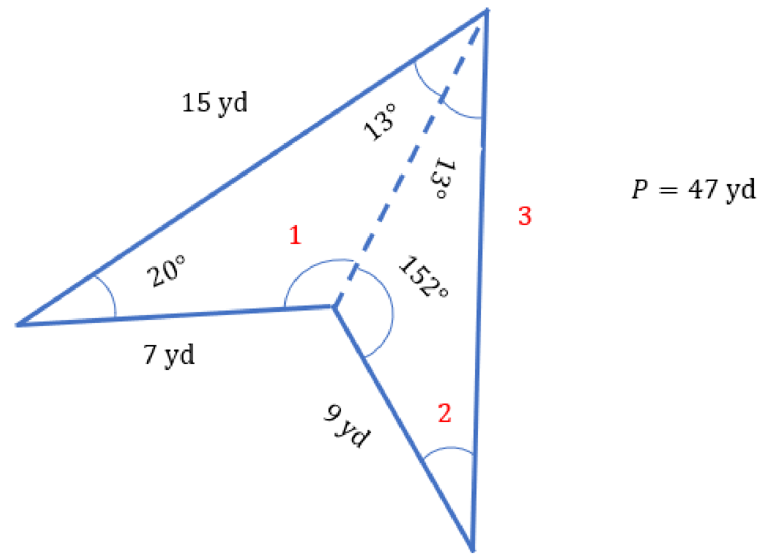
Name:____________________________

Date:_____________________________

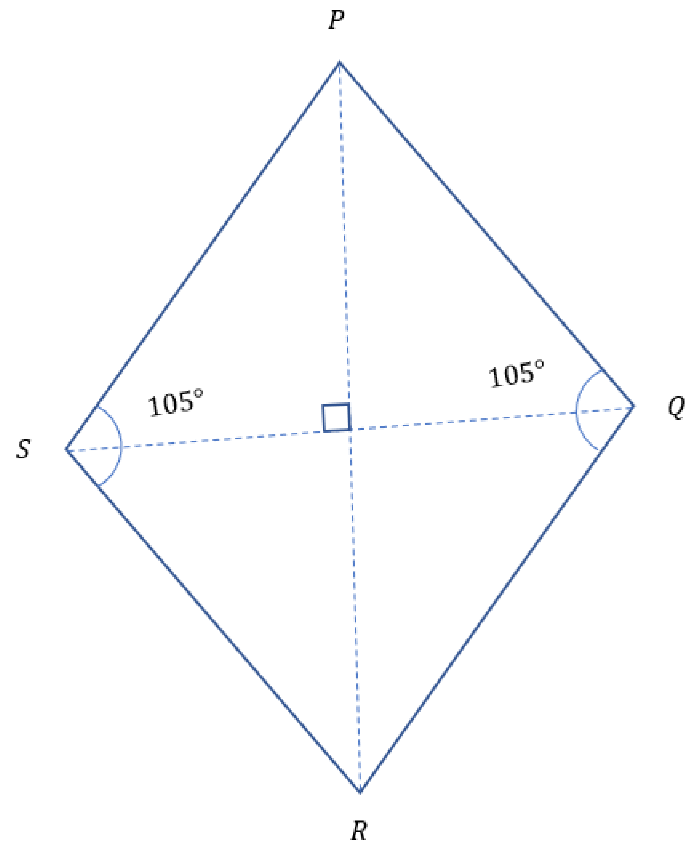
1. In the figure shown below, indicate whether or not the legs of the triangle are congruent.



2. The following image depicts a quadrilateral with a perimeter of 15 ft. Using the given angle measures and the value of the perimeter, find the missing sides and angles.



3. The figure below depicts rhombus $PQRS$.



(a) If each side of the rhombus was perpendicular to one another, then each interior angle would have a measure of 90°. What shape would rhombus $PQRS$ be?

(b) What is the measure of $\angle SPQ$?

(c) State whether or not the following statement is true:

The diagonals are perpendicular ($PR \perp SQ$)

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Manuscript 1031

Solving Logic Problems with a Twist

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Solving Logic Problems with a Twist

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Review of Symbolic Logic

Aristotle, Greek philosopher and scientist, was a pupil of Plato and asserted that any logical argument is reducible to two premises and a conclusion (Gullberg, 1997). According to classical (Aristotelian) logic, arguments are governed by three fundamental laws: the principle of identity, the principle of the excluded middle, and the principle of contradiction. The first law states that *a thing is itself*, i.e., $A = A$. According to the principle of the excluded middle, a proposition is prescribed a discrete truth value (true or false, but there exists no value in between). Lastly, the principle of contradiction asserts that a *single* truth value can be assigned to a statement. A proposition must either be true or false, A or $not\ A$ (Gullberg, 1997).

The intimate relationship between logic and mathematics was established by German mathematician and philosopher G.W. von Leibniz (Gullberg, 1997). Leibniz is recognized as the founder of symbolic logic and for his discovery of a *lingua universalis* – a language where logical errors would be equivalent to errors in mathematics. Furthermore, English mathematician and logician George Boole published the *Mathematical Analysis of Logic* (1847) (Gullberg, 1997). In this work, Boole devised a system of algebra to express logical relations. In 1854, Boole expanded on his algebra in *An Investigation into the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, eventually resulting in the extension of Boolean algebra into electronics (applications in digital circuits) (Gullberg, 1997). Finally, Gottlob Frege is acknowledged as the founder of modern symbolic logic (Gullberg, 1997). In 1879, his contributions to predicate logic were acknowledged in his publication *Begriffsschrift: Eine der aritmetischen nachgebildete Formelschprache des reinen Denkens* ("Concept-Script: A Formal Language of Pure Thought on the Pattern of Arithmetic.") (Gullberg, 1997).

Propositions are statements that can be assigned truth values. As mentioned previously, truth values constitute a binary system: the two possible truth values are T or F. In the context of Boolean algebra, these truth values may be referred to as Boolean constants, where 1 denotes true and 0 denotes false. A tilde ~ is applied in order to negate a proposition; hence, $\sim p$ is read "not p". The variable p represents any arbitrary proposition. For instance, suppose we let p denote "the ball is red". Negating this statement results in "the ball is not red", or $\sim p$. Note that "is red" is referred to as the predicate, i.e., the portion of the statement describing the state or condition of the noun.

In addition to negation, another logic symbol is the universal quantifier, $\forall$, denoting "for every" or "for all" (Gullberg, 1997). As an example of its application, suppose you wish to state that a polynomial $P(x)$ is defined for all real numbers, i.e., its domain is $(-\infty, \infty)$. An equivalent symbolic representation is provided by $P(x)$ is defined $\forall x \in \mathbb{R}$ ("$P(x)$ is defined for every x belonging to the set of real numbers"). Similarly, the existential quantifier $\exists$ also describes *how* *much* or *how many* values can belong to a set. For instance, $\exists x \in \mathbb{R}, 2x - 5 > 7$ ("there exists at least one x belonging to the set of real numbers, such that the proposition (inequality) is valid") (Gullberg, 1997). Moreover, the conjunction $\wedge$ denotes "and" and is analogous to multiplication in Boolean algebra. For instance, in symbolic logic $A \wedge B$ reads "A and B", and is denoted as $A \otimes B$, $A \cdot B$, or simply AB in Boolean algebra. Furthermore, the disjunction $\vee$ indicates "or". The proposition $A \vee B$ reads "A or B", and is denoted $A \oplus B$, or simply $A + B$ in Boolean algebra. Note that subtraction does not exist in logic; therefore, division does not exist (since it is repeated subtraction). On the other hand, because multiplication is compounded addition…

$$x \oplus x = x \quad and \quad x \otimes x = x.$$

These properties will be discussed in the following sections.

Furthermore, propositions can take the form of "if-then" statements: $A \Rightarrow B$, read "A implies B" or "if A then B". In this logical expression, the term preceding the implication symbol is referred to as the antecedent (hypothesis), whereas the subsequent term is the conclusion (Gullberg, 1997). Note that the expression $A \Rightarrow B$ is false *only* when A is true and B is false (Gullberg, 1997). We have discussed the procedure for combining quantifiers or unary connectives to propositions (operators that have a single statement as their operand, such as $\forall, \exists, \sim$), combining two propositions via binary logical connectives (such as $\oplus, \Rightarrow$, etc.), and now we can apply this foundational knowledge to form even larger logical expressions consisting of many logical connectives. For instance, we can form expressions such as

$\sim A \Rightarrow \sim B$, which contains two unary connectives (the negations acting on each term) and a binary connective (the implication acts on both $\sim A$ and $\sim B$). Suppose A represents the proposition "two lines intersect at point, P" and B denotes "the ordered pair describing P is a solution to a system of equations". This suggests that the statement $A \Rightarrow B$ indicates "if two lines intersect at point P, then the ordered pair describing P is a solution to a system of equations". The inverse of this statement is found by negating both sides (and it is logically equivalent). In other words, $\sim A \Rightarrow \sim B$ has the same meaning as $A \Rightarrow B$.

Finally, these complex propositions can constitute the premises and conclusions of our logical arguments. Now that we are familiar with the notations, we can practice constructing a few basic arguments. According to Palmer (2011), an argument is *valid* if the premises logically entail the conclusion. In other words, there is no flaw in the *form* of the argument or its reasoning. A common logical argument takes the form:

1. $\boldsymbol{A \Rightarrow B}$
2. **A**

$\therefore \boldsymbol{B}$

A logical argument of this form is called *modus ponendo ponens* (Palmer, 2011). As a concrete example:

1. **If I choose door 1 then I am free**
2. **I choose door 1**

$\therefore$ **I am free.**

An inference of the form:

1. $\boldsymbol{A \Rightarrow B}$
2. **~B**

$\therefore \boldsymbol{\sim A}$

is called *modus tollendo tollens* (Palmer, 2011). Notice how the conclusion in premise 1 is negated in premise 2. The conclusion of the argument suggests that the antecedent is negated.

Finally, it is important to discuss the meaning of a *tautology*. "A tautology is a proposition in which the clause following the word "is" or the word "are" (i.e., what is called in grammar the copula) repeats in some form or other all or part of the clause before the copula"(Palmer, 2011). All equations and definitions are tautological since they state that two expressions of different forms are equal. For instance, "a sister is a female sibling", "a fork is a utensil", "$x^2 - 4 = (x + 2)(x - 2)$". The statement "a derivative is the slope of a tangent line to a point on a curve" is not a tautological proposition. This is because information is not repeated in the expression equated to "derivative"; rather, additional information is provided.

Five Additional Logic Problem Examples and Applications

1. Digital Circuits – Finding Boolean Output Expression for Logic Gate

The following truth table depicts three inputs $A, B,$ and C and an output X for a digital circuit in an electronic device. A zero (0) denotes a component that is switched "off" in the circuit, whereas a one (1) indicates that an element is switched "on". Find (a) the logic expression for the output, X, (b) a simplified logic expression for the output X,
(c) design a simpler circuit, and (d) construct a Karnaugh map (K-map).

A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
1	0	0	0
0 $\bar{A}$	1 B	1 C	1
1 A	0 $\bar{B}$	1 C	1
1	1	0	0
1 A	1 B	1 C	1

(a) <u>Boolean expression for output, *X*</u>

The output occurs when it has a truth value of 1; therefore, each term in our expression will constitute a row in the truth table that satisfies this condition. Referring to the given truth table (above), start by circling each 1 in the column for the output, X. In each row, if an input variable is assigned a truth value of 0, then this indicates a negated input. For instance, if A is designated a truth value of 0, then our expression for the output states, $\bar{A}$.

Therefore, the output is given by…

$\boldsymbol{X} = \color{red}{\boldsymbol{\bar{A}BC}} \oplus \color{blue}{\boldsymbol{A\bar{B}C}} \oplus \color{green}{\boldsymbol{ABC}}.$

(b) Simplify the expression for the output

$$X = \bar{A}BC \oplus \underline{A}\bar{B}\underline{C} \oplus \underline{A}B\underline{C}$$

$$= \bar{A}BC \oplus AC(\underline{\bar{B}} + \underline{B})$$

$$= \bar{A}BC \oplus AC$$

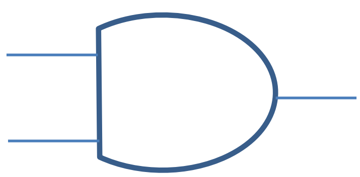
$$= (\bar{A}B \oplus A)C$$

$$= (A \oplus B)C\ .$$

Since AC appears in both term 2 and term 3, we can factor out this expression. Although tempting, do not combine any like terms in the manner: $A + A = 2A$, for example. Recall, this is Boolean algebra, not traditional algebra. The notation $\oplus$ serves as a reminder.

Note: Line 2 was simplified via the property: $\bar{B} + B = 1$.
Line 4 was simplified via the property $\bar{A}B \oplus A = A \oplus B$.

(c) Design a circuit based off the simplified expression for the output:
In the schematic, note that…

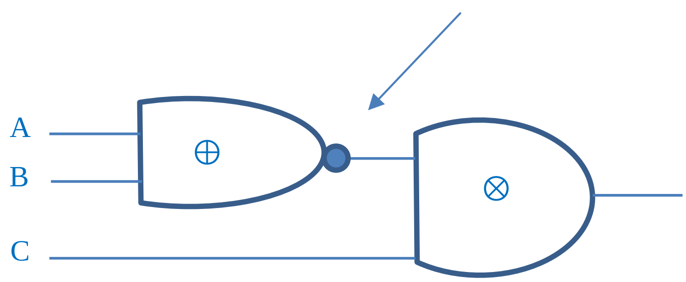


$= and = \otimes$

$= or = \oplus$

Therefore, the circuit corresponding to the output $X = (A \oplus B)C$ is…

At this point, your (binary) logical operation $\oplus$ acts on inputs A and B, outputting $A \oplus B$.



This **gate** is also analogous to a **function machine**, where the inputs are $A \oplus B$ and C, the function (operation) is the $\otimes$, and the output is $(A \oplus B) \otimes C$.

(d) Construct a Karnaugh map:
In applications such as circuits and computer science, a Karnaugh map is a graphical representation of the simplified Boolean expression extracted from the truth table. Using a graphical method as an alternative solution is a wonderful problem-solving strategy, because it verifies that our logical (Boolean) expression for the output is correct. To construct a Karnaugh map, reference the given truth table and count the number of rows. Since the truth table consists of 8 rows, our Karnaugh map is comprised of 2 columns and 4 rows. Each entry in the Karnaugh map is an output (the value for X) corresponding to each input.

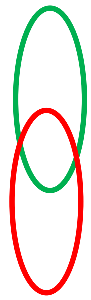
A B \ C	0	1
0 0	0	0
0 1	0	1
1 1	0	1
1 0	0	1

The entries for the inputs, A and B, are arranged in a memorized pattern known as *gray code* (or reflected binary code) in computer science. Similar to a truth table, it is simply all possible combinations of ones and zeros. As mentioned, the entries in each cell denote a value of the output variable. For instance, the first cell reads "the output is $X = 0$ when $A = 0, B = 0$, and $C = 0$ in the truth table. In column 2, row 2 the Karnaugh map reads "$X = 1$ when $A = 0, B = 1$, and $C = 1$ in the truth table.

Although the Karnaugh map is complete, let's extract the simplified Boolean expression for the output from this graphical representation. Previously, we painstakingly derived this logical expression from the truth table itself. After doing so, we manipulated this complex logical expression and simplified it via properties of Boolean algebra. Utilizing our Karnaugh map, let's verify that we obtain the *same* expression for X. The next step is to circle groups of 1's because each term in the expression for the output corresponds to $X = 1$, otherwise the device is switched off. You can only form groups of 1 by circling entries horizontally or vertically (not diagonally).

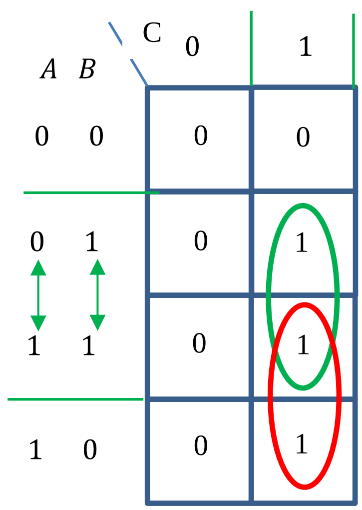
Furthermore, the objective is to enclose as many entries as possible in each group; otherwise, you will have more terms than necessary in your logical expression (potentially repeating or redundant terms). As a consequence, you would not derive the expression for output in its simplest form, which would require you to simplify it via Boolean algebra, defeating the purpose of this method. Again, the benefit of this method is that you find the output in its simplest form without having to do any algebra, otherwise it becomes the same procedure as finding the expression directly from the truth table itself. Another rule pertaining to grouping 1's in the Karnaugh map is that you can only circle 2^n entries. For instance, a group can consist of $2^0 = 1$ ones, $2^1 = 2$ ones, $2^2 = 4$ ones, $2^3 = 8$ ones, and so on.

A B \ C	0	1
0 0	0	0
0 1	0	1



Notice how there are two entries enclosed by the green (upper) contour. We cannot enclose three entries because it violates the rule that a group must contain 2^n elements. Moreover, notice how we could have encased one entry; however, our goal is to optimize the number of Boolean constants enclosed in a group. The next group (red (lower) contour) could also enclose a single element, but again we wish to maximize the number of members belonging to a set. Therefore, a property of the Karnaugh map is that we can **overlap** groups, such that members of one group can also belong to another set.

The final law governing Karnaugh maps is that if an input variable changes its value in the group's row or column, then it is not included in the output expression.



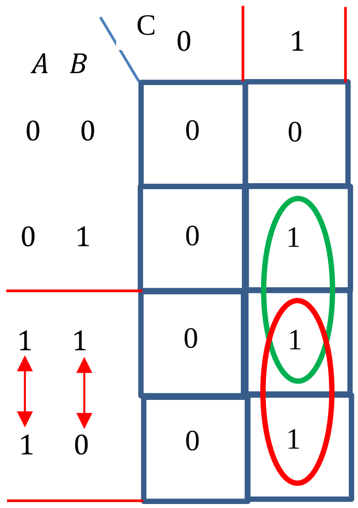
Rows where green group resides (rows 2 and 3): $A: 0 \rightarrow 1$
$B: 1 \rightarrow 1$

Column where green group resides (column 2): $C: 1$

Therefore, since the input variable A alternates values, it represents terms such as $A \oplus \overline{A}$, which is equal to one. We do not include these additional variables in our expression because their value is unity. On the other hand, B is constant in the rows of interest, i.e., it does not fluctuate between truth values; hence, it denotes an expression of the form $B \oplus B = B$. Note the benefit of the notation $\oplus$ opposed to "+" is that it serves as a reminder that like terms are not combined the same way as they are in traditional algebra, where $B + B = 2B$. However, either notation can be utilized; it is simply a matter of preference. Again, this is because $\forall B \in \mathbb{R}, B + B = 2B$ in algebra, whereas $\forall B \epsilon \{0, 1\}, B \oplus B = B$ in Boolean algebra. Finally, the green loop only occupies a single column, and therefore, C is constant at the value 1.

Putting this all together, we see that one term in the output must be CB.

Looking at the red loop, we do a similar procedure:



Rows where red group resides (rows 3 and 4): $A: 1 \rightarrow 1$
$B: 1 \rightarrow 0$

Column where red group resides (column 2): $C: 1$

Therefore, term two in the output is given by… CA.

Note that B is not included, because its value alternates (suggesting that its expression takes the form that is equal to 1). Finally, combining the term from the green group with the term in the red group gives the logic expression for the entire truth table. Now for the moment of truth…

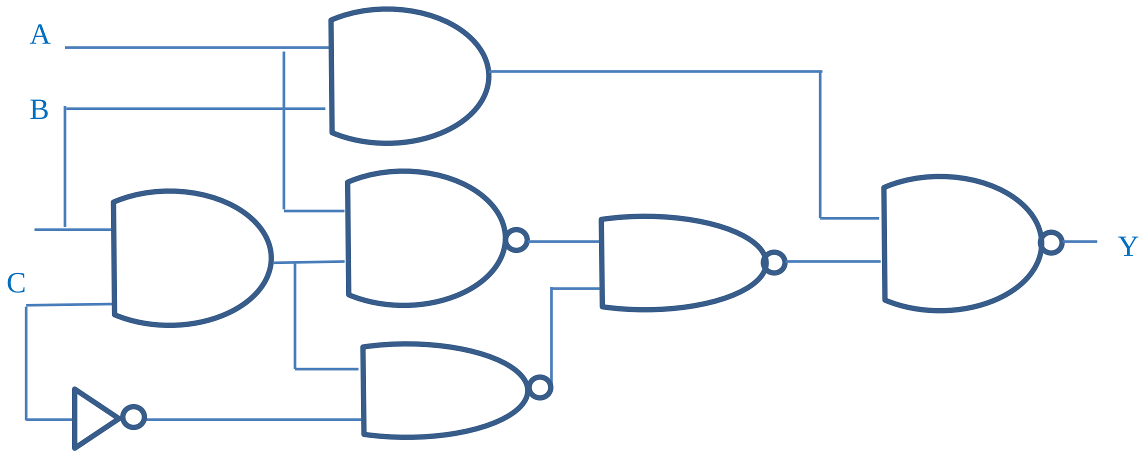
$$\boxed{X = CB \oplus CA = C(B \oplus A)}$$

The proof is complete. The logical expression derived from the Karnaugh map is equivalent to the expression directly obtained from the truth table and brute force Boolean algebra.

2. Digital Circuits – Finding the Boolean Output Expression Directly from Logic Gates

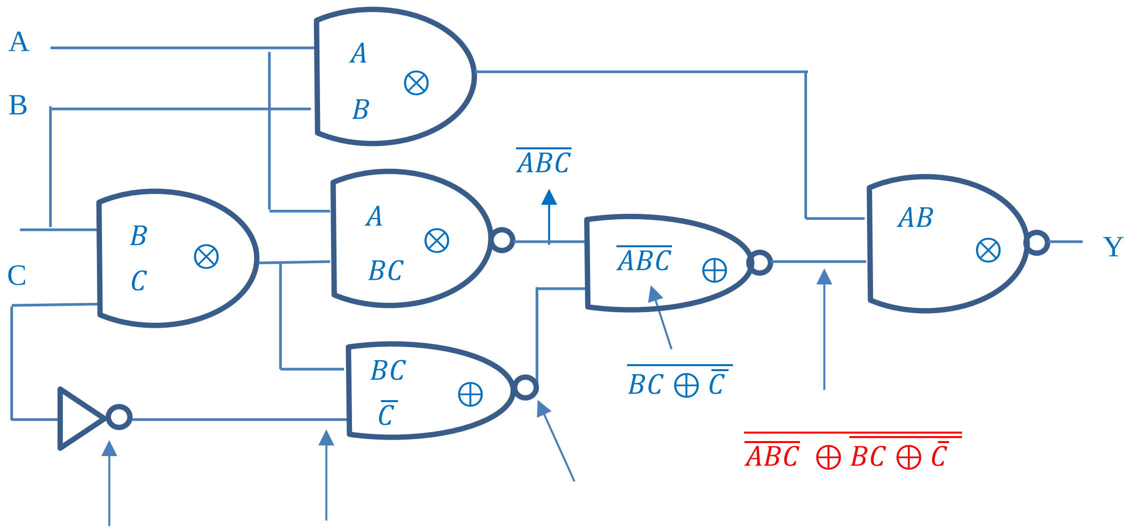
Suppose the following schematic represents an electronic device with inputs $A, B,$ and C. Find the Boolean output Y corresponding to the logic gate diagram.

Note: After obtaining the logical expression, reduce it to a simpler expression using properties of Boolean algebra. This is a great exercise that also allows you to alternate between logic circuits, truth tables, and logic expressions.



As mentioned in the previous problem, each logic gate is analogous to a function machine, processing and operating on our inputs. The AND gate multiplies two inputs, whereas the OR gate adds two operands. A great strategy is to write "OR", "AND", "⊕", or "⊗" on a respective schematic symbol, allowing you to navigate through the diagram with ease. Furthermore, some new symbols have been introduced in this new schematic: is just another symbol for NOT and it can be tacked onto any other symbol in the diagram. The symbol is the NOT gate and it simply negates an input value and does nothing more. If the input is A then this function machine outputs $\sim A$, or in the notation of Boolean logic, $\bar{A}$.

Notice as each input traverses the logic circuit; it receives additional processing (it is operated on more and more by various binary and unary operators). As you navigate through the circuit, the Boolean expression will become increasingly complex as it is outputted from logic gate to logic gate (function machine to function machine). Our goal is to ultimately determine the final product, or the output generated at the very end of the circuit. Inevitably, we will simplify this complex expression into a compact form. Assigning the (Boolean) logic symbol to each component in the schematic…



C is inputted into this component. This logic gate outputs $\bar{C}$.

$\bar{C}$ is flowing through from the other side. Although it is the output of the previous logic gate, $\bar{C}$ is the input for the proceeding OR gate.

The expression is becoming rather complex. We could keep building our logical expression and simplify once it exits the final logic gate. Alternatively, we could have simplified the logical expression each step along the way. Before inputting the red expression into the final AND gate, let's simplify the sum…

..$\overline{\overline{ABC} \oplus \overline{BC \oplus \bar{C}}}$

$= \overline{(\overline{ABC})} \otimes \overline{(\overline{BC \oplus \bar{C}})}$

Applying De Morgan's Theorem: The bar covering the entire expression is split between two terms (where the first $\oplus$ appears). The first $\oplus$ then becomes $\otimes$.

At this point, there are numerous ways to simplify the logical statement. By inspecting the expression above, one can recognize the form $BC \oplus \bar{C} = B \oplus C$. Also notice, we applied the property $\bar{\bar{x}} = x$ (this has the same meaning as $\sim(\sim x) = x$). This illustrates the purpose of De Morgan's Theorem, allowing us to cancel some NOT operators…

$(ABC) \otimes (B \oplus C)$

$= ABCB \oplus ABCC$

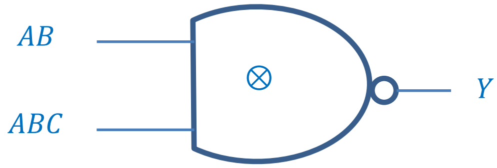
Simplify according to the property $BB = B$ and $CC = C$.

$= ABC \oplus ABC$

Apply the property X + X = X

$= ABC.$

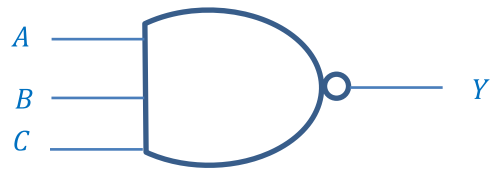
Therefore, $\overline{ABC} \oplus \overline{BC \oplus \bar{C}} = ABC$ is one of the inputs in the FINAL logic gate. Recall, the schematic representing the final logic gate was…



Performing the logical operation of the AND gate…$\overline{AB \otimes ABC} = Y$

$$\boxed{\overline{ABC} = Y.}$$

The entire logical output of that complicated circuit was reduced to a single term, where each of the three input variables occur once in the expression. Likewise, the simplified schematic corresponding to the original circuit is…



In the first example, we were given a truth table and then we converted it into a complex logical statement. After doing so, we simplified the compound proposition into a simpler expression via Boolean algebra. Finally, we proved that the simple expression derived from algebraic manipulations was correct, via a graphical solution utilizing Karnaugh maps. In the second application problem, we saw how propositions are analogous to input variables. Each logic gate is analogous to a function machine, and as the input traverses the circuit, more operations are performed on these logical statements, until they become complex propositions. Our next few examples depict truth tables from pure discrete mathematics (outside the context of electronics and computer science). We focused on methods of *finding* expressions for logical statements (from truth table to logical statement or from logic gate to logical statement). In our next example, we are *given* a logical statement and required to determine its corresponding truth table.

3. Constructing a Truth Table from a Given Logical Statement

Suppose you are given the proposition, $p \rightarrow \sim(p \wedge q)$. Construct a truth table to model this given situation.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \rightarrow \sim(p \wedge q)$
T	T			

T	F			
F	T			
F	F			

We start by writing all possible combinations of T and F (this is analogous to the gray code in the computer science example: all possible combinations of 1's and 0's). The first two columns are the propositions without any logical connectives. Then, we depict the expression inside the parentheses in the next column, and gradually continue to work our way out until we reach the desired statement.

The statement $p \land q$ has a truth value of T only when *both* p and q are true. For instance, in row 3, q possesses a truth value of false; hence, the entire statement $p \land q$ is false. Filling in the third column results in…

p	q	$p \land q$	$\sim(p \land q)$	$p \to \sim(p \land q)$
T	T	T	F	
T	F	F	T	
F	T	F	T	
F	F	F	T	

The next column negates our previous column. This led to the truth values in blue. Finally, the last column contains the implication arrow. Regardless of the truth value of the antecedent, if the conclusion is true, then the entire statement $p \to \sim(p \land q)$ is true. If the conclusion is false, then the entire statement is false. Comparing our first column to our blue column…

p	q	$p \land q$	$\sim(p \land q)$	$p \to \sim(p \land q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

The truth table is complete. The statement is false only in the first row of the truth table.

4. Using logic to answer quantitative problems

Let's assume there are 4 students in the classroom: students $A, B, C,$ and D. The teacher administers writing utensils so each student can complete their assignment. Each student in the class received; 4, 5, or 6 pencils. Suppose students that receive 4 or 6 pencils always tell the truth, whereas students administered 5 pencils never tell the truth. When asked to report the total number of pencils the group received, each student stated the following numbers:

18	20	19	17
A	B	C	D

Given that three of the four students are lying, i.e., they received 5 pencils, which student is the honest student?

There is only one honest student, since the given problem stated that three students received 5 pencils. This suggests that a total of $5(3) = 15$ pencils were distributed to dishonest students.

Subtracting 15 pencils from each student, we can determine which student received either 4 or 6:

$$A: 18 - 15 = 3$$

$$B: 20 - 15 = 5$$

$$C: 19 - 15 = 4$$

$$D: 17 - 15 = 2$$

Therefore, the only student with 4 or 6 pencils remaining, is student, C.

5. **Suppose you are given the following statement:** It is summer and there is rain.
(a) Express the given statement symbolically
(b) Apply De Morgan's Theorem to negate this statement
(c) Construct a truth table for the statement expressed in part (b).

The given statement consists of two propositions p and q combined by a single connective (and).

$$\underbrace{\text{It is summer}}_{p} \text{ and } \underbrace{\text{there is rain.}}_{q}$$

(a) Expressing the statement symbolically: $p \wedge q$

(b) Negating the statement using De Morgan's Theorem: $\sim(p \wedge q) = \sim p \vee \sim q$

Note that we applied the same process in one of the application examples.
In De Morgan's Theorem, the sign alternates from multiplication to addition or vice versa.

Hence, the statement reads "it is not summer or there is no rain".

(c) Devise a truth table for the statement $= \sim p \vee \sim q$, i.e., "it is not summer or there is no rain". Since there are 2 statements, our truth table will consist of $2^2 = 4$ rows:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	$\boldsymbol{\sim p \vee \sim q}$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Solution to the Given Problem:

Suppose a prisoner must make a choice between two doors. One door leads to freedom and the other door is booby-trapped so that it leads to death. The doors are labeled as follows:

Door 1: This door leads to freedom and the other door leads to death.

Door 2: One of these doors leads to freedom and the other of these doors leads to death.

If exactly one of the signs is true, which door should the prisoner choose? Give reasons for your answer. Provide a detailed explanation of how you approach the problem. Use at least two different models for your solutions.

Solution (Model) 1.

Convert the verbal (written) statements to symbolic logic. In order to create a logical expression, we will start by assigning arbitrary variables to each proposition.

Let ***p*** denote ***Door 1 leads to freedom*** and let ***q*** denote ***Door 2 leads to death.*** Using this notation, the first proposition becomes…

$$p \wedge q$$

Statement two can also be expressed symbolically…

$$(p \wedge q) \vee \sim(p \wedge q)$$

Using De Morgan's theorem, we can replace $\sim(p \wedge q)$ with $\sim p \vee \sim q$…

$$(p \wedge q) \vee (\sim p \vee \sim q)$$

Perhaps this forms a complex logical expression with some redundancies. If this is the case, we can form a compound proposition, simplify it, and then translate the meaning of the compact logical statement. Combining propositions 1 and 2 gives…

$$(p \wedge q) \vee (p \wedge q) \vee (\sim p \vee \sim q).$$

This expression simplifies to… $p \wedge q$

Therefore, the compound proposition formed from both doors reduces to the statement on door 1. Thus, door 1 leads to freedom and door 2 leads to death.

Solution (Model) 2.

Door 1: This door leads to freedom and the other door leads to death.

Door 2: One of these doors leads to freedom and the other of these doors leads to death.

Door 1 is p and door 2 is q. Using the formula if p, then q, the only conclusion that makes the statement true is, if door 1 leads to freedom, then door 2 does not lead to freedom i.e. death.

p	q	$p \wedge \sim q$ $[(p \wedge \sim q) \vee (\sim p \wedge q)]$
F	*F*	*F*
F	*T*	*F*
T	*F*	*T*
T	*T*	*F*

Summary

The crossroads of logic and mathematics breeds critically thinking students that have the confidence and independent working skills to tackle any problem they come across. In this artifact several applications of logic to common problems have been applied in a way that would allow teachers to use the examples in their classrooms in a wide range of grade levels. More specifically Boolean Algebra and Truth Tables have been making their way into middle and high school mathematics classrooms with the nations shift from just being able to compute to understanding on a cognitive level why you solved the problem the way you did. In United States schools the standardized testing is becoming increasingly rigorous as the years go by and mathematics concepts that used to be reserved for higher education students is now considered everyday math. "Mathematical knowledge and the ability to solve quantifiable problems and utilize critical thinking skills enhance the abilities of students to think and make decisions (Su, 2016). Applying logic to common math problems in low pressure classroom setting prepares students and interests them in higher level mathematics applications. The goal of every teacher is to expand the mind of and inspire their students to go farther in the subject matter than they ever thought possible and teaching them these logic skills and applications will help them get there.

Boolean Logic is a method of algebra that explores three words commonly known as Boolean Operators: "Or," "And," and "Not". Boolean logic is at its core the basic idea that any and all statements are either true or false. The use of Boolean Logic allows students to think on a critical level of more complex mathematical applications, this allows students to categorize whatever information is given and use deductive and/or inductive reasoning skills to arrive at an answer to any problem. The use of simple "and", "or," and "not," Venn diagram problems are found commonly in Algebra 1 classrooms and can be found as early as Pre-Algebra in some middle school classrooms. Teaching students the terminology and logic behind these types of problems gives them a concrete foundation to apply these skills in their higher-level mathematics classes. It is essentially giving them a foundation for their future probability courses and has many other applications. Introducing students to "out of the box" ways to approach simple problems with pique their interest and encourage them research independently and be excited to share what they have learned with the class.

One of the tools of Boolean Algebra that was used in this artifact is a truth table. Truth tables are mathematical tables that organize the true or false outcomes to compound statements. Each statement is usually represented by a variable, most commonly p, q, and r, and each statement has its own corresponding column in the table that contains the likely truth values. Truth tables could be introduced in high school geometry when they are learning reasoning skills. "By using inductive and deductive reasoning and questioning, while adhering to the Blooms Taxonomy, students learn mathematical concepts and solve mathematical problems and also recognize the extent to which reasoning applies to mathematics and how their critical thinking pathway can facilitate decision making and strategic planning (Su, 2016)." Even though logic is considered higher order thinking and complex, when teachers introduce students to these concepts as soon as possible and constantly over time, they become simple. Promoting the highest levels of Bloom's Taxonomy will enable teachers to ensure all students are mastering the content while building habits that will aid them in their chosen career.

Use in the Classroom

Teachers are under incredible pressure to teach content and prepare students for standardized tests, but content is not the only thing students need to know in order to be successful test takers as well as productive citizens. This missing link is critical thinking. Students are not only asked to regurgitate facts and processes on standardized tests, they need to be able to coherently describe their thinking. Students need to be able to examine questions logically in order to fully understand what is being asked of them, and in return answer in a comprehensive way (Harlin 2013). This is a skill that students need to be taught, which leads to the question how exactly as teachers can you foster logical thinkers?

A great way to build logical thinkers is through logic puzzles. The wonderful thing about logic puzzles is they can be adapted to any grade level. The next problem occurs when teachers try to decide where to put these puzzles into their content packed schedule. A few suggestions would be as a warm-up, center, or even homework activity. Logical puzzles can also be used to build collaboration skills when assigned to a group to solve. Another great thing about these puzzles is they do not need to be linked to your current content, which allows them to be used as early finisher activities or just as tasks to keep students engaged in the learning.

Along with many benefits of using logic puzzles there are a few mind fields that educators can encounter when using them. Teachers who have ELLs in their classrooms might notice that the language barrier will hinder the student's ability to complete the assignment. This is due to the fact that logic problems require a command over the English language that these students may not possess. Another problem teacher may encounter is the amount of time that more sophisticated problems will require. A way to combat both of these problems is allowing students to work in groups (Bouchard n.d.).

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